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Matching Markets with Signals

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Abstract

A costless signaling mechanism has been proposed as a device to improve welfare in decentralized two-sided matching markets. An example of such an environment is a job market for new Ph.D. economists. We study a market game of incomplete information between firms and workers and show that costless signaling is actually harmful in some matching markets. Specifically, if agents have very similar preferences, signaling lessens the total number of matches and the welfare of firms, as well as it affects ambiguously the welfare of workers. These results run contrary to previous findings that costless signaling facilitates match formation.

1 Introduction.

Signaling as an actual mechanism design instrument was first implemented by the Ad Hoc Committee[1] of American Economic Association (AEA) in December 2006 to facilitate match formation in the job market of new Ph.D. economists. This market begins in the early Fall each year when economic departments advertise for open faculty positions and graduate students nearing completion of their dissertations apply to these positions according to their preferences. Additionally, each student has an opportunity to send two signals to two departments prior to the market[2]. Each signal states only that the student has indicated her interests to some department. Only the faculty of the chosen department knows that the student has sent her signal. The main part of the market happens afterwards when departments invite students for interviews and finally choose the best candidates, whom they make job offers. However, each department can interview only a small portion of students, which creates congestion. The decision which candidates to invite for interviews is a strategic one. An average department does not want to find itself interviewing the same candidates who are being interviewed by the elite departments.

The Ad Hoc Committee introduced the signals in order to alleviate the congestion on the interviewing stage. Signaling is essentially a costless communication, or cheap talk. There is no penalty attaches for lying, and claims do not directly affect payoffs. Therefore, the signals can only enlarge the set of equilibria. Crawford and Sobel (1982) show that cheap talk can be credible in an equilibrium if parties have common interests. Moreover, costless communication leads to new equilibria that are Pareto-superior to the one without communication. Therefore, one may conjecture that cheap talk should be also beneficial for decentralized matching markets. Roth (2008b) also suggests that the limited number of signals can credibly transmit information about students' preference, which could help to reduce the coordination failures faced by the market participants and facilitate better match formation (see also "Signaling for Interviews in the Economics Job Market"[3] for the

[1] The Ad Hoc Committee was established in 2005 in order to develop ways to facilitate the market for new economists. Its members are Alvin E. Roth (chair), John Cawley, Philip Levine, Muriel Niederle, and John Siegfried.

[2] This mechanism is implemented via AEA website http://www.aeaweb.org/joe/signal/.

[3] The document has been created by the Ad Hoc Committee (AEA) to provide advice for participants of Job Market for new Economists, http://www.aeaweb.org/joe/signal/signaling.pdf.

discussion). Recently, Coles and Niederle (2007) obtain results that support this intuition[4]. They consider one-to-one matching market between firms and workers, whose preferences are ex-ante uniformly distributed over the range of all possible preference order lists (preferences are equally likely). Each worker can send only one signal to some firm. They show that the introduction of signals increases the expected number of matches and the welfare of workers in equilibria[5].

We consider a more realistic assumption (than a uniform distribution) on agents' preferences and show that signals impede match formation in our environment. Though signals transmit information about agents' types truthfully, they also introduce information asymmetry. The information asymmetry facilitates coordination failures that decrease the expected number of matches and the welfare of agents. This decrease of agents' welfare in new cheap talk equilibria is in the line of Farrell and Gibbons (1989)'s results, though it differs in its intuition[6]. Costless communication in their two-agent bargaining model gives the buyer an opportunity to pretend to have a lower value and the seller an opportunity to pretend to have a higher value (compare to truthful information transmission in our model). This enhances their bargaining positions at the cost of the risk of no trade. New cheap talk equilibria are characterized by both less trade and a reduction in the expected gains from trade.

We analyze one-to-one matching market between workers and firms in this paper. We examine an environment when workers have almost aligned preferences. Each worker has either "typical" commonly known preferences with probability close to one or "atypical" preferences taken from some distribution with complementary probability close to zero. The preferences of workers are ex-ante independently distributed. Firms have identical and commonly known preferences over workers. We consider a decentralized matching game with three stages. First, each worker chooses some firm, which she sends her signal to. Each worker can send at most one signal[7]. Workers send signals simultaneously. Only firms that

[4] See also Parendo (2007) for related work.

[5] Coles and Niederle (2007)'s analysis is limited by the assumption that each firm can see only the signal from its most preferred worker. Kushnir (2008) relaxes this assumption and extends their results to the case of many-to-one matching market with many signals. He also shows explicitly that the introduction of signals ambiguously affects the welfare of firms.

[6] We are thankful for Lones Smith who drew our attention to this comparison.

[7] A worker can abstain from sending any signal.

receive signals observe them. Second, firms make decisions about job offers taking into account signals received at the first stage. Each firm can make only one offer. Finally, each worker chooses some offer to accept among available offers. Each worker can accept at most one offer.

We show that if firms respond to signals in this environment, i.e. treat signals informatively, the introduction of signals decreases the expected number of matches and the welfare of firms. The effect of signals on the welfare of workers is ambiguous. Intuitively, they help workers with "atypical" preferences to obtain better matches, but at the same time, they deprive some workers with "typical" preferences of their matches. Overall, our analysis reveals two important roles of signals. On the one hand, signals reduce coordination failures because they transmit previously unavailable information about workers' preferences. On the other hand, signals introduce information asymmetry. They transmit information about the preferences of students to the limited number of firms, leaving the other firms uninformed. This information asymmetry facilitates coordination failures.

Let us illustrate why signals can facilitate coordination failures by a simple example with three firms and three workers. The firms rank the workers in the same order (w_1, w_2, w_3), i.e. they strictly prefer worker w_1 to worker w_2 to worker w_3. Each worker's preference is either "typical" (f_1, f_2, f_3) with probability $1-\varepsilon$, $\varepsilon << 1$ or "atypical" with complementary probability ε[8]. "Atypical" preferences are uniformly distributed among all possible preference order lists. All workers are acceptable to all firms and vice versa.

If signals are not allowed, the only possible match in an equilibrium is the assortative match, in which each firm is matched to the corresponding worker. If signals are allowed, we consider the following equilibrium strategies of agents. Each worker with "typical" preferences sends her signal to the corresponding firm (worker w_i sends her signal to firm f_i). Each worker with "atypical" preferences send her signal to the best firm worse or equal to the corresponding one (according to "typical" preferences). Each firm makes her offer to a worker better or equal to the corresponding one only if it receives a signal from her. Each firm ignores all signals from workers worse than the corresponding one. If a firm receives no signals, it makes an offer to the best worker worse than the corresponding one.

[8]See section 2 for the exact assumption on ε.

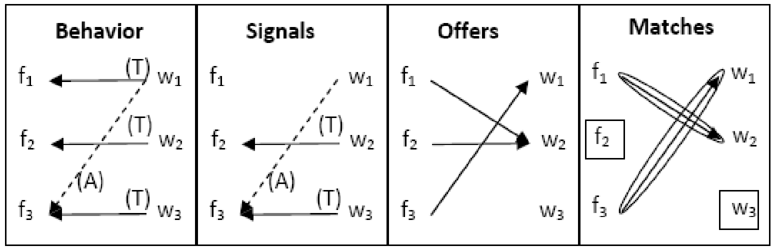

Figure I.

Let us consider the realization of preference profiles when only worker w_1 is "atypical" and firm f_3 is her favorite firm. Worker w_2 and worker w_3 are "typical". Then, worker w_1 sends her signal to firm f_3. Worker w_2 and worker w_3 send their signals to firm f_2 and firm f_3 correspondingly. Firm f_3 makes her offer to worker w_1, and firm f_1 anticipates that worker w_1 is "atypical" and makes her offer to worker w_2. However, the coordination failure arises because firm f_2 has no information about worker w_1's type and cannot anticipate firm f_1's behavior. Firm f_2 makes also her offer to worker w_2, and it eventually ends up unmatched because worker w_2 accepts firm $f_1's$ offer. Thus, the number of matches for some realization of preferences is fewer than the number of matches when the signals are not allowed. Therefore, the expected number of matches is also fewer.

A substantial part of the literature on two-sided matching markets studies centralized markets that employ deferred acceptance algorithm proposed by Gale and Shapley (1962). The outcome of this algorithm is a "stable" matching, in which no agents is matched to an unacceptable agent of the other side of the market, and no pair of agents is unmatched if it prefers to be matched. Centralized clearinghouses organized around the deferred acceptance algorithm can deliver thickness to the market, help deal with the congestion, and make it safe to participate (Roth (2008b)). These desirable properties have allowed some failed markets to be successfully reorganized. Roth (2008a) and Roth and Sotomayor (1990) present an excellent overview of the main theoretical accomplishments in this area.

However, many labor markets are decentralized or at least preceded by decentralized opportunities for participants to match. Therefore, the analysis of decentralized matching markets outcomes and devices that facilitate match formation for these markets appear to be an important issue. Alcalde and Romero-Medina (2000) propose a simple mechanism in form of a two-stage game that implements stable matches[9]. Pais (2006) models decentralized matching markets by means of a sequential game where firms are randomly given the opportunity to make job offers. She shows that every stable matching can be reached as the outcome of an equilibrium play of the game. Niederle and Yariv (2008) is one of the few papers that study decentralized matching markets under incomplete information. They show that strong assumptions are required for the existence of equilibrium strategies that yield a stable outcome in the presence of uncertainty and frictions.

Finally, we want to note that this paper does not analyze search for matches. Agents usually need to perform costly search to locate a better partner in decentralized matching markets. Contrary to search literature (see Chade and Smith (2006), Lee and Schwarz (2007), Kircher (2008)), we assume that agents perfectly know the payoffs from their matches.

The paper proceeds as follows. Section 2 outlines our general model. Our main result that costless signaling is actually harmful for some matching markets is presented in section 3. Section 4 derives two roles that signals play in matching markets: transmit information and facilitate asymmetry. Finally, section 5 concludes and outlines some directions for further research.

2 Model.

We consider a two-sided matching market with N workers and M firms, $N \geq M$. The set of workers and the set of firms are denoted as W and F correspondingly. W and F include the empty set $\{\varnothing\}$. Each worker w has a strict preference order θ_w over firms. Similarly, each firm f has a strict preference order θ_f over workers. Θ_W and Θ_F are the set of all possible workers' and firms' preference orders. Each agent knows only her/its preference order list and has some ex-ante common believes about the other agents' preferences. We assume that agents' preferences are independently distributed and denote the joint distribution of agents' types as $g(\theta)$, where $\theta \in (\Theta_W)^N \times (\Theta_F)^M$.

Though we mainly analyze ordinal utilities of agents, we employ the equilibrium concept

[9] See also Alcalde (1996), Alcalde et al. (1998), and Haeringer and Wooders (2008).

which demands the specification of cardinal utilities. As well, we need cardinal utilities in order to analyze welfare properties of equilibria. Each agent a has cardinal utility compatible with her/its preference ranking θ_a[10]. If worker w with ranking θ_w is matched with firm f, she receives utility $u_w(f, \theta_w)$. Similarly, if firm f with ranking θ_f is matched with worker w, it receives utility $u_f(w, \theta_f)$.

To make our exposition more comprehensible we assume that cardinal utility from being matched with an agent on the kth position in one's ranking is the same across agents. We denote this cardinal utility as δ_k. Additionally, agent's cardinal utility from being unmatched is normalized to zero. We also assume that there is no worker which firms do not want to hire, and there is no worker who prefers being unemployed to being matched with some firm, i.e. for any k, $\delta_k > 0$.

Our main goal is to model the influence of signals on the congested markets. In order to accomplish it, we consider one period model of workers and firms. Each worker can send at most one signal and accept at most one offer. Then, each firm has only one vacant position and can make at most one offer[11]. The timing of the game is the following.

1. *Agents' preferences are realized. Workers send signals to firms simultaneously. Only firms who receive signals observe them.*

2. *Firms make offers to workers simultaneously. Using knowledge of the received signals each firm makes its offer to some worker. A firm can make an offer to any worker independently on whether it receives a signal from this worker or not.*

3. *Workers choose offers to accept.*

We restrict our analysis to pure strategies[12]. A strategy of worker w is a duple $v_w = (v_w^1, v_w^2)$ that represents her decisions on the first and third stages. A strategy of a worker on the first stage is to choose a firm she sends her signal to, $v_w^1 : \Theta_W \to F$. A strategy of

[10] We employ cardinal utilities compatible with ordinal ranking similar to Bogomolnaia and Moulin (2001).

[11] In practice, some firms should rationally make several offers, anticipating that some workers probably reject their offers. We do not model this firms' decision.

[12] Mixed strategy does not add much to our analysis, though they make comprehension of our results more involved.

a worker on the last stage is to choose an offer among available to her, $v_w^2 : \Theta_W \times 2^F \to F$, where $2^F = \{h : h \subset F\}$.

A strategy of firm f is its decision on the second stage. Based on its preferences and a set of received signals firm f chooses which worker it makes an offer to, $v_f : \Theta_F \times 2^W \to W$, where $2^W = \{h : h \subset W\}$.

For a given strategy profile of agents $v = (v_w, v_f)$ and realized agents' types $\theta \in (\Theta_W)^N \times (\Theta_F)^M$ one can determine the final matching and agents' utilities. We denote the utility of agent a given a strategy profile v and a profile of types θ as $u_a(v, \theta)$. In the same manner, the interim expected payoff of worker w with preferences θ_w from strategy v_w when the other agents follow a strategy profile v_{-w} equals

$$u_w(v_w|v_{-w}, \theta_w) = \sum_{\theta_{-w}} g(\theta_{-w}) u_w((v_w, v_{-w}), (\theta_w, \theta_{-w}))$$

where $g(\theta_{-w})$ is the joint distribution of all agents except worker w. A firm f interim expected payoff given its preferences θ_f, a subset of received signals $h \subset W$, believes $\mu_f(\cdot|h)$, and others agents' strategy profile v_{-f} is

$$u_f(v_f|v_{-f}, h, \theta_f) = \sum_{\theta} \mu_f(\theta|h) u_f(v_f, v_{-f}, \theta)$$

We employ the concept of perfect Bayesian Equilibrium for multi-stage games with observed actions and incomplete information in order to solve the game(see Fudenberg and Tirole (1991)).

Definition 1. *A strategy profile (v_w^*, v_f^*) and posterior beliefs $\mu_f(\cdot|h)$ for each firm f and each subset of workers $h \subset W$ is a perfect Bayesian Equilibrium if:*

$$\forall w \in W,\ \theta_w \in \Theta_W : v_w^1(\theta_w) \in \arg\max_{\alpha} u_w(\alpha|v_{-w}^*, \theta_w)$$
$$\forall f \in F,\ h \subset W,\ \theta_f \in \Theta_F : v_f(h, \theta_f) \in \arg\max_{\beta} u_f(\beta|v_{-f}^*, h, \theta_f)$$
$$\forall w \in W,\ \theta_w \in \Theta_W,\ h' \subset F : v_w^2(h', \theta_w) \in \arg\max_{\gamma \in h'} u_w(\gamma, \theta_w)$$

Also, for any $f \in F$ and $h \subset W$, $\mu_f(\theta|h) = \prod_{w_i} \mu_f(\theta_{w_i}|h)$, and believes are defined using

Bayesian rule whenever possible.

We present our main results in the next section. We consider a specification of the above model in which the preferences of agents are almost aligned and show that the signals have a negative influence on matching formation in this case.

3 Almost aligned preferences.

This section analyzes the case when agents' preferences are almost aligned. First, we describe the type of agents' preferences we analyze and introduce some definitions. Then, we analyze the benchmark model when signals are not allowed. Afterwards, we proceed with the analysis of the model with signals. In this case, there are two types of pure strategy perfect Bayesian equilibria: *babbling* and *informative*. All firms ignore signals in the former one, which leads to the same matching as in the case when signals are not allowed. However, if some firms respond to signals, which happens in an informative equilibrium, the introduction of signals changes the matching formation. In the end of the section, we discuss the roles of signals in match formation and their implication on the welfare of market participants.

We consider an environment when each worker can be of two types: "typical" or "atypical". A "typical" worker is denoted as $w(T)$ and she has a fixed preference list θ_0, the same for all typical workers. We enumerate firms according preference θ_0, i.e. f_1 is the best firm, f_2 is the second best, etc. An "atypical" worker is denoted as $w(A)$ and she has a preference list θ_w^A, which is ex-ante distributed according to some distribution $A(\Theta_W)$. Each worker is ex-ante "typical" with probability $1-\varepsilon$ and "atypical" with probability ε.

$$\theta_w = \begin{cases} \theta_0 \text{ if } \gamma_w = 0 \\ \theta_w^A \text{ if } \gamma_w = 1 \end{cases}$$

where γ_w, $w \in W$, are independent and identically distributed across workers. Also, $\gamma_s \in \{0,1\}$ and $P(\gamma_s = 1) = \varepsilon$. We assume that ε is small.

Assumption E. $0 < \varepsilon < \min(\min_{j,l} \frac{|\delta_j - \delta_l|}{l\delta_l}, \min_l \frac{\delta_M}{\delta_1 + l\delta_l})$

We also assume that the distribution of "atypical" preferences, $A(\Theta)$, has a full support, i.e. each firm can be the top firm of an "atypical" worker with some positive probability.

Assumption A. $A(\Theta)$ has a full support: for any $f \in F$ and any $w \in W$ $\Pr(f = \max_{\theta_w}\{f' : f' \in F\}) > 0$.

We consider a model, where firms have the same fixed preferences list θ_f[13]. Similar to firms, we also enumerate workers according to this preference list. To avoid unnecessary notation, we omit, henceforth, the dependence of firms' strategies on their preferences, $v_f(h, \theta_f) \equiv v_f(h)$.

Even the signals are voluntary in our model, they still can play negative role and draw away the offers of firms. In order to eliminate such equilibria, we assume that if firm f makes an offer to worker w when it does not receive her signal, firm f makes an offer to worker w when it receives her signal[14].

Assumption PRS (Positive Role of Signals). *For any firm $f \in F$ and any worker $w \in W$ and any $h \subset W$, $w \notin h$, if $v_f(h) = w$ then $v_f(h \cup w) = w$.*

Now we introduce a couple of new notations that are helpful in our further discussion. We call a subset of workers $h \subset W$ as *reached for firm f when agents follow strategy profile v* if ex-ante probability that only workers from set h send their signals to firm f strictly more than zero.

Definition 2. *A subset of workers $h \subset W$ is reached for firm f when agents follow strategy profile v if*

$$\Pr_f(h) = \sum\nolimits_{\theta} g(\theta) \prod\nolimits_{w \in h} I_{v_w(\theta_w)=f} \prod\nolimits_{w' \notin h} (1 - I_{v_{w'}(\theta_{w'})=f}) > 0$$

where $I_{v_w(\theta_w)=f} = \begin{cases} 1 \text{ if } v_w(\theta_w) = f \\ 0 \textit{ otherwise} \end{cases}.$

[13] We believe that our results are still valid for the model in which the preferences of firms are almost aligned.

[14] See example A1in Appendix for an example of an equilibrium when assumption PRS is violated in Appendix.

We call firm f responds to worker w's signal if her signal changes the strategy of firm f with positive probability.

Definition 3. *Firm f responds to worker w's signal if there exists a subset of workers h, $w \notin h$, such that both h and $h \cup w$ are reached for firm f, and $v_f(h) \neq v_f(h \cup w)$.*

Let us consider an environment when workers cannot send signals. Then, the outlined above model is a static game of incomplete information. Therefore, a perfect Bayesian equilibrium coincides with a Bayesian equilibrium and agents' believes are irrelevant.

If signals are not allowed and ε is small, the dominant strategy of firm f_1 is to make an offer to worker w_1. The second top firm anticipates that worker w_1 is likely to accept firm f_1's offer. Hence, the dominant strategy of firm f_2 is to make an offer to worker w_2. Sequentially, the dominant strategy of each firm is to make an offer to the corresponding worker.

Proposition 1 (No signaling equilibrium) *If assumption E holds and signals are not allowed then the only equilibrium match is the assortative match.*

Now, we analyze the set of equilibria in the matching market with signals. We distinguish two kinds of equilibria.

Definition 4.

- *An equilibrium is babbling if no firm responds to any signal.*

- *An equilibrium is informative if there is at least one firm that responds to some worker's signal.*

We proceed with the characterization of equilibria in the following two subsections.

3.1 Babbling equilibria

This section analyzes equilibria when firms do not respond to signals. We first show a result, which we will further frequently refer to. It states that each firm believes that each worker has "typical" preferences with probability more than $1-\varepsilon$ either when it receives her signal or it does not receive her signal.

Proposition 2 *For any worker $w \in W$, any firm $f \in F$, and any set of workers $h \subset W$ either $\mu_f(\theta_w = \theta_0 | h \cup w) \geq 1-\varepsilon$ or $\mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1-\varepsilon$. Similarly, either $\mu_f(\theta_w \neq \theta_0 | h \cup w) \leq \varepsilon$ or $\mu_f(\theta_w \neq \theta_0 | h \backslash w) \leq \varepsilon$.*

Proof.

Let us denote the probability that worker $w(A)$ and worker $w(T)$ send their signals to firm f as α_A and α_T correspondingly. Then, if worker w sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A > 0$, we derive its beliefs using Bayesian rule

$$\begin{cases} \mu_f(\theta_w = \theta_0 | h \cup w) = \frac{(1-\varepsilon)\alpha_T}{(1-\varepsilon)\alpha_T + \varepsilon\alpha_A} \\ \mu_f(\theta_w = \theta_0 | h \backslash w) = \frac{(1-\varepsilon)(1-\alpha_T)}{(1-\varepsilon)(1-\alpha_T) + \varepsilon(1-\alpha_A)} \end{cases}$$

One can verify that

$$\begin{cases} \mu_f(\theta_w = \theta_0 | h \cup w) \geq 1-\varepsilon \quad \Leftrightarrow \quad \alpha_T \geq \alpha_A \\ \mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1-\varepsilon \quad \Leftrightarrow \quad \alpha_T \leq \alpha_A \end{cases}$$

Hence, either $\mu_f(\theta_w = \theta_0 | h \cup w) \geq 1-\varepsilon$ or $\mu_f(\theta_w = \theta_0 | h \backslash w) \geq 1-\varepsilon$.

If worker w never sends her signal to firm f, $(1-\varepsilon)\alpha_T + \varepsilon\alpha_A = 0$, firm f's believes are $\mu_f(\theta_w = \theta_0 | h \backslash w) = 1-\varepsilon$ and $\mu_f(\theta_w = \theta_0 | h \cup w)$ is arbitrary. The second statement follows from the first one. □

Now we are ready to characterize the set of possible matches in a babbling equilibrium. If ε is small (assumption E) and firms do not response to signals, signals play no role in equilibria. Hence, the only possible match in a babbling equilibrium is the assortative match as in our benchmark case.

Theorem 1 *The only possible match in a babbling equilibrium is the assortative match.*

Proof.

Assume that no firm responds to any signal. Let us assume that there exists a realization of agents' preferences such that firm f_1 is matched to some worker w_i, $i > 1$, in an equilibrium. Hence, there exists $h \subset W$, reached in the equilibrium, such that $v_{f_1}(h) = w_i$. If $w_1 \in h$ firm f_1's belief equals $\mu_{f_1}(\theta_{w_1} = \theta_0|h) < 1 - \varepsilon$, otherwise firm f_1 should make her offer to worker w_1. Hence, $\mu_{f_1}(\theta_{w_1} = \theta_0|h\backslash w_1) > 1 - \varepsilon$ (see proposition 2) and worker w_1 secures firm f_1's offer in the equilibrium if she does not send her signal to it and firm f_1 receives signals only from set $h\backslash w_1$ of workers, $v_{f_1}(h\backslash w_1) = w_1$. Firm f_1 never responds to signals. Therefore, this is possible only if $h\backslash w_1$ is not reached for firm f_1 in the equilibrium. However, the strategies of workers are not correlated. Hence, if set h is reached and set $h\backslash w_1$ is not reached for firm f_1, worker w_1 always sends her signal to firm f_1. However, this means that $\mu_{f_1}(\theta_{w_1} = \theta_0|h) = 1 - \varepsilon$. Hence, firm f_1's strategy is suboptimal. Firm f_1's payoff is maximized if it makes its offer to worker w_1. One may similarly show a contradiction if $w_1 \notin h$.

The other possibility that firm f_1 is not matched to worker w_1 in an equilibrium is that firm f_1 is unmatched. Let us assume that this is the case for some profile of preferences in an equilibrium. Hence, worker w_1 prefers some firm f_k, $k > 1$, to firm f_1 for this profile of preferences, i.e. she is "atypical". Therefore, there exists $h' \subset W$, reached for firm f_k in the equilibrium, such that $v_{f_k}(h') = w_1$. If set $h'\backslash w_1$ is not reached for firm f_k in the equilibrium, its belief about worker w_1 is $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h') = \varepsilon$, which contradicts the rationality of firm f_k's behavior[15]. Therefore, set $h'\backslash w_1$ is reached for firm f_k in the equilibrium. Firm f_k should make an offer to worker w_1, i.e. $v_{f_k}(h) = v_{f_k}(h\backslash w_1) = w_1$, because firm f_k does nor respond to signals. Hence, its beliefs according to proposition 2 are either $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h) \leq \varepsilon$ or $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h\backslash w_1) \leq \varepsilon$. Assume the former one, $\mu_{f_k}(\theta_{w_1} \neq \theta_0|h) \leq \varepsilon$[16]. If firm f_k makes her offer to worker w_1, $v_{f_k}(h\backslash w_1) = w_1$, it should believe that $\mu_{f_k}(\theta_{w_k} = \theta_0|h) < 1-\varepsilon$, otherwise firm f_k plays suboptimal strategy. Hence, firm f_k's belief is $\mu_{f_k}(\theta_{w_k} = \theta_0|h \cup w_k) \geq 1 - \varepsilon$

[15] The argument about unreachable sets is omitted further. If either $h' \cup w_1$ or $h'\backslash w_1$ is not reached for firm f_k, either worker w always sends her signal to firm f_k or worker w_1 never sends her signal to firm f_k correspondingly. Hence, the below arguments follow trivially.

[16] If we assume the latter one, the argument is similar.

(if $h \cup w_k$ is not reached for firm f_k in the equilibrium the argument is trivial) and firm f_k's strategies are $v_{f_k}(h \cup w_k) = v_{f_k}(h) = w_1$. However, this strategy cannot be optimal, because firm f_k's payoff from making an offer to worker w_k, when worker w_k sends her signal to it, equals at least $(1-\varepsilon)\delta_k$. Taking into account assumption E, we obtain that $(1-\varepsilon)\delta_k > \varepsilon\delta_1$. Contradiction.

We argued above that the best feasible worker for firm f_2 is worker w_2. Hence, using similar speculations one may show that firm f_2 can be matched only with worker w_2 in a babbling equilibrium. Therefore, the only feasible match in a babbling equilibrium is the assortative match. It is a routine to show that the assortative match can be supported by some system of beliefs in a perfect Bayesian equilibrium. □

3.2 Informative equilibria

This subsection considers equilibria in which at least one firm responds to some worker's signal. There is a great multiplicity of such equilibria. For example, each firm can believe that it is the worst program in worker's preferences if it receives her signal. Taking into account this belief this worker rationally never sends her signal to this firm in equilibrium. Therefore, these believes are off-equilibrium path, and they are consistent with the notion of perfect Bayesian equilibrium.

In order to make our analysis more focused we consider in this subsection the equilibria when each firm responds to signals from any worker better or equal to the corresponding one.[17]. According to assumption A, each firm is the best firm for each worker with positive probability. Hence, each worker prefers this firm to all other firms with positive probability. Hence, if she could use her signal to receive its offer, she would use her signal. In the same time, each firm would prefer the match with each worker better than the corresponding one. The assumption that each firm responds to signals from any worker better or equal to the corresponding one guarantees that each firm should make its offer to such workers if it does not receive a better signal[18].

[17] This refinement makes the results more comprehensible. All the results are still true if there is enough programs that responds to students' signals.

[18] See proposition A2 in Appendix for more detailed explanation.

We postpone the explanation of patterns of agents' behavior in informative equilibria to Appendix. It appears that a firm never makes its offer to a worker better than the corresponding one if it does not receive a signal from her in an equilibrium. In addition, if a firm responds to some worker's signal and her signal is the best signal it receives, the firm makes its offer to this worker. We concentrate here only on the discussion of our main result about existence and uniqueness of equilibria.

Let us denote the set of workers worse or equal to worker w_j as $\Delta_j = \{w_1, ..., w_j\} \subset W$. Then, there is a unique equilibrium such that each firm responds to all signals from workers better or equal to the corresponding one.

Theorem 2 *If each firm responds to all signals from workers better or equal to the corresponding one, the set of strategies*

- $v_{w_i}(\theta_{w_i}) = \max_{\theta_{w_i}}(f' \in F : f' \preceq_{\theta_0} f_i)$

- $v_{f_j}(h) = \begin{cases} \max_{\theta_{f_j}}(w : w \in h) \ \textit{if}\ h \cap \Delta_j \neq 0 \\ w_{j+1} \qquad \textit{if}\ h \cap \Delta_j = 0 \end{cases}$

and set of firms' beliefs consistent with agents' strategies constitute a unique equilibrium.

Basically, each worker sends her signal to the best firm (according to her preferences) among the firms worse or equal to the corresponding one (according to "typical" preference). If firm f_j receives at least one signal from corresponding worker w_j or better one, it makes an offer to the best such a worker, otherwise it makes an offer to worker w_{j+1}.

The above result completes our description of agents' behavior in equilibria. If we do not assume that each firm responds to signals from any worker better or equal to the corresponding one, there is a great multiplicity of equilibria in our model. However, it is still possible to derive the welfare comparison and evaluate the role of signals for all equilibria. We proceed with this analysis in the next subsection.

3.3 Welfare properties of equilibria

We evaluate the effect of signals from ex-ante perspective. We mainly use the following quantitative characteristics: the expected number of matches, the expected total welfare of firms and the expected total welfare of workers. However, theorem 4 provides also some results for the effect of signals on the individual firms.

Let us denote the ex-post number of matches when agents follow profile of strategies v, and the realized profile of preferences is $\theta \in \Theta_W \times \Theta_F$ as $m(v, \theta)$. The expected number of matches when agents follow profile of strategies v can be expressed as

$$EM(v) = \sum_{\theta} g(\theta) m(v(\theta), \theta)$$

Similarly, the expected total welfare of workers and firms equal

$$\begin{aligned} EW_{\text{firm}}(v) &= \sum_f \sum_{\theta} g(\theta) Pr_f(h_f(\theta)) u_f(v_f(\theta) | v_{-f}(\theta), h_f(\theta), \theta_f) \\ EW_{\text{worker}}(v) &= \sum_w \sum_{\theta} g(\theta) u_w(v_w(\theta_w) | v_{-w}(\theta_{-w}), \theta_w) \end{aligned}$$

correspondingly.

Proposition 1 shows that the expected number of matches in any "no signaling" equilibrium equals M (under assumption that $N \geq M$), which is the maximum one. Hence, it is impossible that the expected number of matches in any informative equilibrium exceeds the expected number of matches in any "no signaling" equilibrium. Example 1 and example 2 demonstrate the case of equality and strict inequality for this welfare criterion.

The preferences of firms are the same. Firms are also matched to the best workers in any "no signaling" equilibrium. Hence, the decrease in the expected number of matches leads to the decrease of the expected total welfare of firms.

The result about total workers' welfare is not such straightforward and depends on their cardinal utility. The intuition is that signals help "atypical" workers. In the same time, they decrease the expected number of matches, which makes worse some workers. Example 3 illustrates this point.

Example 1 (Equal expected number of matches)

Let us consider the following equilibrium strategies.

- for any $l < M$, $v_{w_l}(\theta_{w_l}) = \max_{\theta_{w_l}}(f : f \in \{f_l, f_{l+1}\})$

- for any $l \geq M$ $v_{w_M}(\theta_0) = f_M$

- if $v_{f_l}(h) = \begin{cases} \max_{\theta_f}(w : w \in h) \text{ if } h \cap \Delta_l \neq 0 \\ w_{l+1} \quad\quad \text{if } h \cap \Delta_l = 0 \end{cases}$

If for any $h \subset W$ and for any $1 \leq l \leq M$, $1 \leq k \leq N$, $k \neq l, l-1$, we denote $\Theta_{f_l} = \{\theta \in \Theta : f_l = \min_\theta(f : f \in F)\}$. Then, firm $f_l's$ off-equilibrium path beliefs are such that for any $h \subset W$ $\mu_{f_l}(\theta_{w_k} \in \Theta_{f_l}|h \cup w_k) = 1$. As one may check, the described strategies constitute an equilibrium. Moreover, the introduction of signals does not lead to the loss of the number of matches. If worker w_l is "atypical" and prefers firm f_{l+1} to firm f_l, she sends her signal to firm f_{l+1}. Firm f_{l+1} makes her offer to worker w_l. In the same time, firm f_l makes her offer to worker w_{l+1}. There is no loss in the number of matches, because firm f_l and firm f_{l+1} exchange their matches. The total welfare of firms neither changes. The expected welfare of workers increases, because they end up with better matches than in any "no signaling" equilibrium.

Example 2 (Fewer expected number of matches)

The fact that the expected number of matches decreases is illustrated perfectly by the example in the introduction. One may extend this example to the case with more workers and firms. We present only figure II to illustrate the idea. The explanation is similar to the one in the introduction.

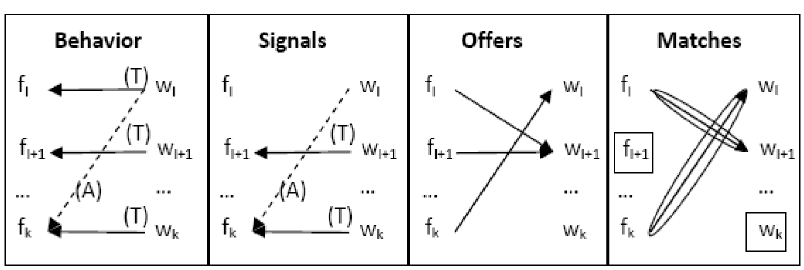

Figure II.

Example 3 (Welfare of workers)

Let us consider three firms and three workers and the uniform distribution of "atypical" preferences $A(\Theta) = U(\Theta)$. Workers' cardinal utilities: $(\delta_1, \delta_2, \delta_3) = (\delta + \lambda, \delta, \delta - \lambda)$. The expected total welfare of workers when signals are not allowed equals

$$EW_{\text{worker}}^{\text{nosignals}} = \sum_{i=1}^{3} \left[(1-\varepsilon)\,\delta_i + \varepsilon \tfrac{1}{3} \sum_{l=1}^{3} \delta_l \right] = 3\delta$$

Let us consider a perfect Bayesian equilibrium, described in theorem 2. One may check that the expected total welfare of workers equals[19]

$$EW_{\text{worker}}^{\text{signals}} = 3\delta + \left(-\tfrac{1}{3}\delta + \tfrac{19}{6}\lambda \right) \varepsilon$$

Hence, the expected total welfare of workers increases if and only if the difference in utilities between adjacent firms is large enough, $\lambda > \frac{2}{19}\delta$. This illustrates that the introduction of signals is beneficial for a matching market according to egalitarian welfare criterion if and only if the decrease in the total number of matches is offset by better matches of "atypical" workers.

The below theorem summarize the results derived above.

Theorem 3 - *The expected number of matches in any informative equilibrium is weakly fewer than in any "no signaling" equilibrium.*

- *The expected total welfare of firms in any informative equilibrium is weakly less than in "no signaling" equilibrium.*

- *The effect of signals on the expected total welfare of workers is ambiguous.*

We have compared an informative and "no signaling" equilibrium using aggregate welfare criteria. The following theorem provides more strict results for individual firms.

[19] The terms ε^2 and ε^3 are ignored.

Theorem 4 *If firm f_{l-1} and at least one firm among $\{f_{l+1}, ..., f_M\}$ responds to worker w_{l-1}'s signal in an informative equilibrium, then*

- *firm f_l's expected number of matches is strictly fewer in any "informative" equilibrium than in any "no signaling" equilibrium.*

  Moreover, if no worker among $\{s_1, ..., s_{l-1}\}$ sends her signal to firm f_l in any "informative" equilibrium, then

- *firm f_l's expected welfare is strictly less in an "informative" equilibrium than in any "no signaling" equilibrium.*

4 Role of signals in matching markets

The previous study by Coles and Niederle (2007) shows that the introduction of signals facilitates match formation. They consider a model similar to the model of this paper except that agents preferences are uniformly distributed, $\theta_s \sim U(\Theta_s)$ and $\theta_p \sim U(\Theta_p)$. They show that the introduction of signals increase the expected number of matches and the welfare of workers[20]. As shown in the previous section, these results, however, highly rely on the assumption that the preferences of agents are uniformly distributed.

If the preferences of workers are almost aligned and the preferences of firms are the same, the introduction of signals decreases the expected number of matches and the expected total welfare of firms. The influence of signals on the welfare of workers is ambiguous. Overall, table I presents the effects from the introduction of the signals for the two different environments: almost aligned and uniform distribution of preferences.

[20] Kushnir (2008) relaxes one of their assumptions and extends their results to the case of many-to-one matching market with many signals. He also shows explicitly that the introduction of signals ambiguously affects the welfare of firms.

Preferences	No signals	EMatches	EW_{worker}	EW_{firm}
Almost aligned	0	−	±	−
Uniform distribution	0	+	+	±

Table I. Almost aligned VS uniform distribution preferences

A natural question is why signals influence matching markets in different ways. We argue that the signals play two different roles: transmit information and facilitate information asymmetry. On the one hand, the introduction of signals helps "atypical" workers to transmit the information about their preferences and locate a better match. On the other hand, signals transmit information to small number of firms, which facilitate information asymmetry. This information asymmetry leads to coordination failures, which decrease the number of matches in the market.

Preferences	Transmit information	Facilitate information asymmetry
Almost aligned	Small	**Large**
Uniform distribution	**Large**	Small

Table II. The roles of signals

When there is ex-ante small amount of information about agents' preferences, the information transmission plays more important role in match formation. This happens when agents' preferences are ex-ante uniformly distributed as in Coles and Niederle (2007). However, when there is almost complete information about agents' preferences (as in this paper), the introduction of signals leads to coordination failures. Overall, the signals play the controversial roles in match formation process. This makes them a less powerful tool than it was previously anticipated.

5 Conclusion

We exemplify an environment when the introduction of signals harms matching markets: lessens the expected number of matches, decreases the welfare of firms, and affects ambiguously the welfare of workers. Based on this example, we argue that the signals play controversial roles in match formation. Though they help to transmit information about participants' preferences, they also facilitate the information asymmetry among them. While the former one reduces coordination failures and facilitates better match formation, the latter one acts in the opposite direction. We leave here, as an open empirical question, what effect dominates in real matching markets.

We consider a one-period model where each worker sends at most one signal and each firm has only one vacant position. However, these assumptions are not crucial for our results. Several (finite) periods would allow firms to secure a better match, but signals would still introduce information asymmetry. If each worker sent several signals, it would transmit information to greater number of firms, but each signal would be less informative. Several vacant positions would make only the preferences of firms more complicated and would not influence the results. Overall, the roles of signals in match formation robust to these modifications.

One interesting question for further theoretical research is to analyze the effect of signals for a larger range of preference distributions. Though an analytical analysis can be a cumbersome for other cases, a computer simulation seems a perfectly suitable tool to accomplish this goal.

Another interesting question is to explore the effect of public signals. Do the signals benefit matching markets if all firms can observe them? One potential problem with public signals is that they may induce some complicated strategic behavior by agents, which may render them useless.

A Appendix

The first two propositions are devoted to the description of agents' strategies in informative equilibria. The first proposition shows that it is too risky for a firm to make an offer to a worker better than the corresponding one if it does not receive a signal from her.

Proposition A1 (Offers to better workers) *If firm f_j does not receive a signal from worker w_i, $i < j$, it does not make an offer to her in an equilibrium.*

Proof.

We proof this proposition by induction. Let us show the validity of the claim for $j = 2$. We consider two possibilities: either worker $w_1(T)$ does not send a signal to firm f_1, $v_{w_1}(\theta_0) \neq f_1$, or she sends a signal to firm f_1, $v_{w_1}(\theta_0) = f_1$.

Worker w_1 employs strategy $v_{w_1}(\theta_0) \neq f_1$ in an equilibrium only if firm f_1 makes its offer to worker w_1 with probability equals to one. Firm f_2 has a chance to be matched with worker w_1 only if she is "atypical". Similar to proposition 2, we denote the probability that worker $w_1(A)$ and worker $w_1(T)$ send a signal to firm f_2 as α_A and α_T correspondingly.

Let us assume that worker $w_1(T)$ sends her signal to firm f_2 in an equilibrium, i.e. $\alpha_T = 1$. If firm f_2 does not receive a signal from worker w_1, it believes that worker w_1 is "atypical". According to assumption PRS, if firm f_2 makes an offer to worker w_1 when it does not receive her signal, it should make an offer to her when it receives her signal. However, firm f_2's believes are such in this case that for any $h \subset W$

$$\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \cup w_1) = \frac{\alpha_A \varepsilon}{\alpha_A \varepsilon + \alpha_T (1-\varepsilon)} \leq \varepsilon$$

Now, if worker $w_1(T)$ does not send her signal to firm f_2, ($\alpha_T = 0$), firm f_2's belief is such that for any $h \subset W$

$$\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \backslash w_1) = \frac{(1-\alpha_A)\varepsilon}{(1-\alpha_A)\varepsilon + (1-\alpha_T)(1-\varepsilon)} \leq \varepsilon$$

For both cases, $\alpha_T = 1$ and $\alpha_T = 0$, firm f_2's payoff from making her offer to worker w_1 is less than $\varepsilon \delta_1$. According to assumption $M \leq N$, firm f_2 can secure a match with some worker w_t, $t > 2$, with probability at least $1 - \varepsilon$. Hence, firm f_2 does not make an offer to worker w_1 in an equilibrium.

If worker w_1 employs strategy $v_{w_1}(\theta_0) = f_1$, similar to the above discussion firm f_2's belief is $\mu_{f_2}(\theta_{w_1} \neq \theta_0 | h \backslash w_1) \leq \varepsilon$. Hence, it is again suboptimal for firm f_2 to make an offer to worker w_1.

We have shown that it is not optimal for firm f_2 to make an offer to a better student if it does not send her signal to it. Let us assume that the claim is valid for any f_j, $j < k$ and show that the claim is valid for firm f_k. Let us consider some worker w_i, $i < k$. Firm f_i makes her offer to workers better than worker w_i with probability less than $\varepsilon\,(i-1)$. In addition, worker w_i is "atypical" with probability ε. Hence, firm f_k can secure a match with worker w_i with probability less than $i\varepsilon$. Assumption E guarantees that ε is small enough that firm f_k's offer to worker w_i is suboptimal. □

Now we assume that each firm responds to a signal from any worker better or equal to the corresponding one. The following proposition shows that a firm makes its offer to some worker better or equal to the corresponding one if the worker's signal is the best signal the firm receives.

Proposition A2 (Response to signals) *Assume that $M > N$, and each firm responds to a signal from any worker better or equal to the corresponding one. Then, for any firm f_j, worker w_i, $i \leq j$, and any $h \subset W$ $v_{f_j}(h \cup w_i \backslash \Delta_{i-1}) = w_i$ in an equilibrium*[21].

Proof.

We proof this statement by induction. Let us consider first firm f_1 and worker w_1. If worker w_1 employs strategy $v_{w_1}(\theta_0) \neq f_1$, for any $h \subset W$ firm f_1's belief is $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \backslash w_1) \geq 1 - \varepsilon$. Hence, firm f_1 always makes an offer to worker w_1, which contradicts to our assumption that it responds to worker w_1's signal. Therefore, the only possible optimal strategy of worker w_1 is $v_{w_1}(\theta_0) = f_1$. In this case, for any $h \subset W$ firm f_1's belief is $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \cup w_1) \geq 1 - \varepsilon$. Hence, firm f_1's the highest expected payoff when it receives

[21] If $M = N$ the claim is still valid with the same assumption for all firms except firm f_M. Firm f_M should respond to a signal from any worker strictly better than the corresponding one.

worker w_1's signal is from making an offer to worker w_1. Hence, for any $h \subset W$, firm f_1's strategy $v_{f_1}(h \cup w_i \backslash \Delta_{i-1}) = w_1$ is optimal.

Assume now that for any $t \leq j < k$, and for any $h \subset W$, firm f_j employs strategy $v_{f_j}(h \cup w_t \backslash \Delta_{t-1}) = w_t$. We prove below that firm f_k's optimal strategy is for any $h \subset W$ and any $i \leq k$, $v_{f_k}(h \cup w_i \backslash \Delta_{i-1}) = w_i$.

Let us consider some $i \leq k$. There are two possibilities: either $v_{w_k}(\theta_0) \neq f_k$ or $v_{w_k}(\theta_0) = f_k$. For the former case, for any $h \subset W$, $\mu_{f_k}(\theta_{w_k} = \theta_0 | h \backslash w_k) \geq 1 - \varepsilon$. Hence, it is optimal for firm f_k to make an offer to worker w_k when it receives no signals from workers from Δ_k, i.e. for any $h' \subset W$ $v_{f_k}(h \backslash w_k \backslash \Delta_{k-1}) = w_k$. Hence, it is also optimal for firm f_k to make an offer to when worker w_k's signal is the best signal it receives, i.e. $v_{f_k}(h \cup w_k \backslash \Delta_{k-1}) = w_k$. In addition, if firm f_k receives a signal from some worker, which belongs to set Δ_k, it makes an offer to it independently on worker w_k's signal according to the induction assumption. Therefore, firm f_k does not respond to signal from w_k. Contradiction.

For the latter case, $v_{w_k}(\theta_0) = f_k$, if firm f_k does not receive a signal from worker w_k, it anticipates that she is "atypical", and it does not make an offer to her in an equilibrium. If firm f_j receives signals from the set $h \cup w_i \backslash \Delta_{i-1}$ (for some $h \subset W$), no other firm f_p, $p \neq j$ and $p > i$, makes its offer to worker w_i according to proposition. The only competing with firm f_j's offer could be the offers from firms in set $\{f_p,\, p < i\}$. However, any firm f_p, $p < i$, could make an offer to worker w_i only if worker w_p is "atypical", which happens with probability ε. Hence, the interim expected payoff for firm f_j from making its offer to worker w_i equals at least $(1-(i-1)\varepsilon)\delta_k$ which the highest interim payoff it could receive. Therefore, firm f_j makes its offer to worker w_i, $v_{f_j}(h \cup w_i \backslash \Delta_{i-1}) = w_i$. $\square$

Proof of theorem 2.

We first prove that the set of strategies, stated in the theorem, constitute an equilibrium. Then, we show that given assumption E, assumption A, and the assumption that all firms respond to a signal from any worker better or equal to the corresponding one, the equilibrium is unique.

Let us show that, if all agents, except firm f_l, follow the strategies, stated in the theorem, firm f_l's strategy is optimal given its believes consistent with the other agents' strategies.

If firm f_l receives a signal from worker w_t, $t < l$, firm f_l believes that itself is the best firm among set $\{f' \in F : f' \preceq_{\theta_0} f_t\}$ for worker w_t. Let us assume that worker w_t is the best worker that sends a signal to firm f_l. Worker w_t does not accepts firm f_l's offer only if she receives an offer form firm f_{t-1}. However, it happens only if worker w_{t-1} is atypical, i.e. with probability less than ε. Hence, firm f_l interim expected payoff from making an offer to worker w_t equals at least $(1-\varepsilon)\,\delta_t$.

Firm f_l's expected payoff from making an offer to a worker better than worker w_t is less than $(1-\varepsilon)\,\delta_t$ according to proposition A1 and assumption E. Firm f_l's expected payoff from making an offer to worker w_k, $k > t$, is less than making an offer to worker w_t, because $(1-\varepsilon)\,\delta_t > \delta_k$. Overall, firm f_l's strategy is optimal.

Let us show that, if all agents, except worker w_t, follow the strategies, stated in the theorem, worker w_t's strategy is optimal. Firm f_t does not make an offer to worker w_t when it receives a signal from a better worker. Therefore, if worker w_t is "typical", her payoff from sending a signal to firm f_t equals at least $[1-(l-1)\varepsilon]\,\delta_t$. If worker w_t does not send her signal to firm f_t it loses her offer. Her payoff from sending a signal to a firm worse than firm f_t is less or equal to δ_{t-1}, which is less than $[1-(l-1)\varepsilon]\,\delta_t$ according to assumption E. There is also no reason for worker w_t to send a signal to a firm better than firm f_t, because this firm does not respond to it. Hence, worker $w_t(T)'$s strategy is optimal. Using similar logic one may show that worker $w_t(A)$'s strategy is also optimal.

Now we show that the above strategies constitute the unique equilibrium. Under assumption that all firms respond to signals from workers better or equal to the corresponding one, proposition A1 and proposition A2 imply that each firm f_l, $l = 1, ..., M$ has to follow the following strategies in equilibrium: for any $h \subset W$, $v_{f_l}(h \backslash \Delta_l) \neq w_l$ and $v_{f_l}(h \cup w_l \backslash \Delta_{l-1}) = w_l$. Straightforwardly, the only worker $w_l(T)$'s optimal strategy is to send her signals to firm f_l, $v_{w_l}(\theta_0) = s_l$, otherwise, firm f_l's does not make an offer to student w_l.

Let us consider firm $f^* = \max_{\theta_{w_l}}(f' \in F : f' \preceq_{\theta_0} f_l)$. It responds to signals from workers better or equal than the corresponding one and firm f^*'s belief about worker w_l's type is

$$\mu_{f^*}(\theta_{w_l} = \theta_0 | h \backslash w_l) \geq 1-\varepsilon \text{ and } \mu_{f^*}(\theta_{w_l} \neq \theta_0 | h \cup w_l) = 1$$

Therefore, if firm f^* does not receive a signal better than worker w_l's one, it's optimal strategy is to make an offer to worker w_l. Taking into account that firm f^* can receive a signal from a better worker with probability less than $(l-1)\varepsilon$, worker $w_l(A)$'s optimal strategy is to send her signals to firm f^* (assumption E). Hence, the strategies, stated in the theorem, constitute a unique equilibrium. □

Proof of theorem 4.

Assume that some firm f_j, $j \geq l+1$, responds to worker w_{l-1}'s signal. We prove the first statement of the theorem in two steps. First, we show that firm f_l makes an offer to worker w_l with positive probability in any equilibrium. Second, we show that firm f_{l-1} also makes an offer to worker w_l with positive probability in any equilibrium. The offer decisions of firm f_l and firm f_{l-1} appear to be uncorrelated and independent on worker w_l type. Therefore, firm f_l is unmatched with positive probability.

We show that firm f_l makes an offer to worker w_l with positive probability in any equilibrium by induction. Let us consider $l = 1$. There are two possibilities: $v_{w_1}(\theta_0) = f_1$ or $v_{w_1}(\theta_0) \neq f_1$. For the former case, for any $h \subset W$ $\mu_{f_1}(\theta_{w_1} = \theta_0 | h \cup w_1) \geq 1 - \varepsilon$. Hence, it is optimal for firm to make an offer to worker w_1 when it receives a signal from her. Worker w_1 is "typical" with positive probability, which means that firm f_1 makes an offer to worker w_1 with positive probability. The latter case can happen in an equilibrium only when firm f_1 always makes an offer to worker w_1 independently on whether it receives a signal from her (assumption PRS).

Let us assume that for any $i < l$, firm f_i makes an offer to worker w_i with positive probability in an equilibrium. Let us show that firm f_l also makes an offer to worker w_l with positive probability. Firm f_l responds to worker w_i's signal, $i < l$, only if $v_{w_i}(\theta_0) \neq f_l$. Hence, firm f_l makes an offer to some worker in set Δ_{l-1} with probability less than one (see also proposition A1).

Assume worker w_l employs strategy $v_{w_l}(\theta_0) = f_l$. Hence, if firm f_l does not receive a signal from a worker better than the corresponding one (or it does not responds to signals it receives) and firm f_l receives a signal from worker w_l, then firm f_l should rationally make an offer to worker w_l.

Assume worker w_l employs strategy $v_{w_l}(\theta_0) \neq f_l$. Firm f_l makes an offer to worker w_i, $i < l$, in an equilibrium with probability equal at most ε. Therefore, firm f_l is not available to worker w_l with probability at most $(l-1)\,\varepsilon$. Assumption E guarantees that worker w_l employs strategy $v_{w_l}(\theta_0) \neq f_l$ in an equilibrium only if firm f_l follows the following strategy: for any $h \subset W$ $v_{f_l}(h \cup w_l \backslash \Delta_{l-1}) = w_l$ (or it employs this strategy if it receives some signals from workers better than the corresponding one, which it does not respond). Hence, firm f_l makes also an offer to worker w_l with positive probability in this case.

Similar to the above speculations firm f_{l-1} does not receive a signal from a worker better than worker w_{l-1} (or it does not responds to better signals it receives) with positive probability. The fact that firm f_{l-1} responds to worker w_{l-1}'s signal guarantees that $v_{f_{l-1}}(h \backslash \Delta_{l-1}) \neq w_{l-1}$. Therefore, if worker w_{l-1} sends to firm f_{l-1} a signal with positive probability, then for any $h \subset W$ firm f_{l-1} employs strategy $v_{f_{l-1}}(h \cup w_l \backslash \Delta_l) = w_l$ in an equilibrium. If worker w_l never sends a signal to firm f_{l-1}, for any $h \subset W$ firm f_{l-1} employs strategy $v_{f_{l-1}}(h \cup w_l \backslash \Delta_l) = w_l$ in an equilibrium. For both cases, firm f_{l-1} makes an offer to worker w_l with positive probability in an equilibrium.

Both firm f_{l-1} and firm f_l make their offers to worker w_l with positive probability. Moreover, firm f_{l-1} makes offer only if it does not receive a signal from worker w_{l-1}. However, if worker w_{l-1} sends her signal to firm f_j, firm f_l has no information about worker w_{l-1} type and cannot predict behavior of firm f_{l-1}. Therefore, both firms make their offers to worker w_l simultaneously with positive probability in an equilibrium, which makes firm f_l sometimes unmatched.

The second statement follows directly from the first one if one considers how firm f_l can benefit from the introduction of signals. It may benefit if it ends up matched with some worker better than its corresponding one. However, this happens according to proposition A1 only if such a worker sends her a signal. Otherwise, it has only disadvantage from the introduction of signals (being unmatched or being match to a worse worker) as was shown above. □

Example A1 (An equilibrium when assumption PRS does not hold.)

Let us consider two firms and two workers. Assume that each "typical" worker has preferences $\theta_0 = (f_1, f_2)$ and each "atypical" worker has preferences $\theta_A = (f_2, f_1)$ with probability equal to one. Firms prefer worker w_1 to worker w_2. Agents employ the following strategies:

- $v_{w_1}(\theta_0) = f_2$, $v_{w_1}(\theta_A) = f_1$

- $v_{w_2}(\theta_0) = f_1$, $v_{w_2}(\theta_A) = f_2$

- for any $h \subset W$ $v_{f_1}(h) = \begin{cases} w_1 \text{ if } w_1 \notin h \\ w_2 \text{ if } w_1 \in h \end{cases}$, $v_{f_2}(h) = \begin{cases} w_1 \text{ if } w_1 \notin h \\ w_2 \text{ if } w_1 \in h \end{cases}$

  Agents' believes are:

- for any $h \subset W$ $\mu_{f_j}(\theta_{w_i} : f_j = \max_{\theta_{w_i}}(f \in F)|h \backslash w_i) = 1$ and $\mu_{f_j}(\theta_{w_i} : f_j = \min_{\theta_{w_i}}(f \in F)|h \cup w_i) = 1$

It is easy to show that the above strategies and the set of beliefs constitute a perfect Bayesian equilibrium. One may extend this example for the environment with more firms and workers.

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