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Under-connected and Over-connected Networks

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Since the seminal contribution of Jackson & Wolinsky 1996 [A Strategic Model of Social and Economic Networks, JET 71, 44-74] it has been widely acknowledged that the formation of social networks exhibits a general conflict between individual strategic behavior and collective outcome. What has not been studied systematically are the sources of inefficiency. We approach this omission by analyzing the role of positive and negative externalities of link formation. This yields general results that relate situations of positive externalities with stable networks that cannot be "too dense" in a well-defined sense, while situations with negative externalities tend to induce "too dense" networks.

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Keywords: Networks, Network Formation, Connections, Game Theory, Externalities, Spillovers, Stability, Efficiency

JEL-Classification: D85, C72, L14

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1 Introduction

The importance of social and economic networks has been widely recognized in economics, as well as in other social sciences. Applications include personal contacts (e.g. Granovetter, 1974), scientific collaborations (e.g. Newman, 2004), trade between countries (e.g. Goyal and Joshi, 2006b), embeddedness of companies (e.g. Uzzi, 1996), and even marriages of ancient trading families (Padgett and Ansell, 1993).

Given the prevalence of network structure in many economic situations, it seems natural to ask how networks change, when agents alter the network structure in order to pursue their goals. It was a major contribution of the economics literature to propose such models based on game theoretic concepts. The first non-cooperative game theoretic approach to two-sided network formation can be found in Myerson (1991). Myerson (1991) proposes a simultaneous move game of network formation, where players announce their desired links non-cooperatively. A link between two players is formed if both players announce it.

Alike in non-cooperative game theory, a central issue in the theory of network formation is the analysis of equilibrium or stability, i.e. a situation where no player wants to change her links. The standard formulation of Nash-equilibrium in the Myerson network formation game has proved to be a non-satisfying concept due to coordination problems.[1] The seminal contribution of Jackson and Wolinsky (1996) solves this problem by introducing a different concept of stability called pairwise stability. In a pairwise stable network no two players want to form a mutual link and no player wants to cut a link unilaterally. This concept of stability is used and has been refined widely in the literature of network formation games.[2] For different models of network formation, Jackson and Wolinsky (1996) characterized pairwise stable and efficient networks by introducing the utilitarian welfare function.[3] They highlight a central problem in strategic network formation: there is a tension between stability and efficiency, meaning that individual interest can be at odds with societal welfare. Since then, there was a flourishing literature on specific situations of strategic network formation of which two small surveys can be found in Jackson (2004) and Goyal and Joshi (2006a). Some models can be found that analyze the formation of directed networks, e.g. Bala and Goyal (2000), and weighted networks, Bloch and Dutta (2005) and Rogers (2006). The various network formation games provide micro-based models and analyze which networks are stable under various notions of stability and which are efficient.

What has not been explicitly studied are the *sources of inefficiency*. The question is particularly, *how* stable networks generally differ from efficient networks? And, *why* individual interest does not always lead to efficient outcomes?

We approach these questions by analyzing the role of externalities, or spillovers, of link formation. Simply put, positive externalities define situations where agents can profit

[1] Any link, that is desired by both players is not necessarily present in Nash-equilibrium if neither player announces it, e.g. the empty network is always an equilibrium.

[2] Among the most well-known refinements are pairwise Nash stability, Bloch and Jackson (2006); unilateral stability, Buskens and Van de Rijt (2005); strong stability, Jackson and Van de Nouweland (2005); and bilateral stability, Goyal and Vega-Redondo (2007).

[3] E.g. the connections model and the co-author model.

(at least do not suffer) from others who form a relationship; while negative externalities mean that they do not benefit from that action. We argue that both types of externalities correspond to natural settings. Network formation games where direct and indirect connections are the source of benefits represent examples for positive externalities. On the other hand, in a context of competition or rival goods, negative externalities occur.

To compare stable and efficient networks in both contexts, we firstly employ and analyze two well-known notions of stability: pairwise stability, as introduced in Jackson and Wolinsky (1996) and pairwise stability with transfers, which stems from a network formation game allowing for transfers.[4] In addition to our main findings, we also elaborate on the relation of these concepts. In analyzing the welfare properties of these stable networks, we use a very general set of welfare functions, which have to satisfy only a monotonicity property.[5] Given a welfare function, we introduce the notion of *over-connected* and *under-connected* networks. In essence, a network is over-connected if welfare can be improved by deleting some links, while a network is under-connected if an addition of links is welfare improving. We show how this notion helps identify the sources of inefficiencies and can be applied to characterizing stable and efficient networks.

The main result for positive externalities is that there is no stable network that can be socially improved by the severance of links. This result is not dependent on the particular shape of the utility functions nor on the degree of homogeneity. We provide some examples taken from literature – among them is the connections model (see Jackson and Wolinsky, 1996) – and illustrate the implications of the result. For negative externalities the tension between stability and efficiency is just the other way round: In various models of network formation, we observe that efficient networks are altered by individuals adding links, a process that leads to stable networks which are "too dense" from a societal point of view. In the context of transfers, we show that no stable network can be socially improved by the addition of links. Without the assumption of transfers, additional insights are won by restricting attention to a large class of network formation models, where the utility function only depends on the number of links all the players have. Among them are the co-author model, firstly introduced in Jackson and Wolinsky (1996), and the model of patent races by Goyal and Joshi (2006a). Furthermore, we derive properties of the utility function that are sufficient for equivalence between pairwise stability with transfers and pairwise stability. In that case, the result of not being under-connected carries over to pairwise stable networks.

Finally, we extend our analysis to a more general framework that is not restricted to undirected and unweighted networks. The framework includes the formation of networks with directed ties, weighted ties, negative ties, and networks with loops (self-links). The first result addresses models of bilateral network formation, in which a condition called "rejection power" is satisfied. This includes all models based on the Myerson link announcement game. We show for positive externalities that a Nash stable network cannot be over-connected. The second result addresses virtually any model of unilateral network formation. It shows for negative externalities (and positive externalities), that

[4] See Bloch and Jackson (2007) for different approaches to network formation allowing for transfers. For a comparison of the two equilibrium concepts see Bloch and Jackson (2006).

[5] The utilitarian welfare function satisfies this notion. For some of the results, we actually need this specific version of a welfare function.

a Nash stable network cannot be under-connected (respectively over-connected). This result, holding for the general class of monotonic welfare functions, is briefly discussed with respect to the models of Bala and Goyal (2000).

The paper is organized as follows: the subsequent section formally defines the model. The implications of positive externalities on the tension of stability and efficiency are shown in Section 3. Section 4 addresses negative externalities and relates the different stability notions. Section 5 presents an extension of our model. More specifically, we verify how our results carry over to a framework that includes the formation of weighted and directed ties. Section 6 concludes.

2 Model and Definitions

Let $N = \{1, ..., n\}$ be a (finite, fixed) set of agents/players, with $n \geq 3$. A network/graph g is a set of unordered pairs, $\{i, j\}$ with $i \neq j \in N$, that represent the bilateral connections in a non-directed graph. Thus, $ij := \{i, j\} \in g$ means that player i and player j are linked in network g. Let g^N be the set of all subsets of N of size two and G be the set of all possible graphs, $G = \{g : g \subseteq g^N\}$.

By $N_i(g)$ we denote the neighbors of player i in network g, $N_i(g) := \{j \in N \mid ij \in g\}$. Similarly, $L_i(g)$ denotes the set of player i's links in g, $L_i(g) := \{ij \in g \mid j \in N\}$. We define $d_i(g) := |L_i(g)| = |N_i(g)|$, as the number of player i's links, called player i's degree.

For each player $i \in N$ a utility function $u_i : G \to \mathbb{R}$ expresses his preferences over the set of possible graphs. $u = (u_1, ...u_n)$ denotes the profile of utility functions. Decisions to form or to sever links typically do not depend on absolute utility, but on marginal changes in utility. Let $mu_i(g, l)$ be the marginal utility of player i of deleting a set of links l currently in network g, that is $mu_i(g, l) := u_i(g) - u_i(g \setminus l)$ for $(l \subseteq g)$. Equivalently, we denote $mu_i(g \cup l, l) := u_i(g \cup l) - u_i(g)$ as the marginal utility of adding the set of links l to network g.

Each (exogenously given) triple (N, G, u) defines a situation of strategic network formation. From the vast literature of network formation, we employ two of the most common stability notions.[6] The first notion is based on a cooperative framework and was introduced by Jackson and Wolinsky (1996).

Definition 1. *A network g is pairwise stable (PS) if no link will be cut by a single player, and no two players want to form a link:*

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \setminus ij)$ *and* $u_j(g) \geq u_j(g \setminus ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

This well-known definition captures the idea that links can be severed by any involved player, whereas the formation of a link requires the consent of both players. Pairwise stability is a basic notion that can be refined in multiple ways (e.g. unilateral stability,

[6] A game theoretic foundation and a comparison of the three notions can be found in Bloch and Jackson (2006).

Buskens and Van de Rijt, 2005; strong stability, Jackson and Van de Nouweland, 2005; or bilateral stability, Goyal and Vega-Redondo, 2007).[7]

The second notion of stability is based on the idea of transfers and can be found in Bloch and Jackson (2007).

Definition 2. *A network g is pairwise stable with transfers (PS^t) if there does not exist any pair of players that can jointly benefit by adding, respectively cutting, their link:*

(i) $\forall ij \in g, \quad u_i(g) + u_j(g) \geq u_i(g \setminus ij) + u_j(g \setminus ij)$ *and*

(ii) $\forall ij \notin g, \quad u_i(g) + u_j(g) \geq u_i(g \cup ij) + u_j(g \cup ij)$.

We denote by $[PS(u)], [PS^t(u)]$ the sets of stable networks for a utility profile u.

While stability tries to answer which networks emerge based on individual preferences, efficiency addresses the evaluation of networks from a societal point of view. To formally capture efficiency, we use a welfare function $w : \mathbb{R}^n \rightarrow \mathbb{R}$ that typically (but not necessarily) is only dependent on the vector of utilities of all players, given a network g. Abusing notation we will also write $w(g)$ instead of $w(u(g))$. The most commonly used version of a welfare function is the utilitarian welfare function, which simply sums up the utility of all players, $w^u(g) = \sum_{i \in N} u_i(g)$. For some of our results, however, an even weaker way of aggregating utility is sufficient. We only require a welfare function to satisfy monotonicity, that is: $u_i(g) \geq u_i(g') \;\; \forall i \in N \quad \Longrightarrow \quad w(u(g)) \geq w(u(g'))$. This assumption is a very intuitive and weak requirement for a welfare function. A welfare function should evaluate a network g at least as high as a network g' if all players $i \in N$ evaluate g at least as high as g'.

Given a welfare function w, we can define efficiency:

Definition 3. *A network g^* is called efficient with respect to the welfare function w if it is a welfare maximizing network, that is $w(g^*) \geq w(g) \;\; \forall g \in G$.*

In many network formation games we observe a general tension between stability and efficiency.[8] Individual interest often conflicts with social welfare. In the following we want to ask under what conditions this tension is observed. Specifically, we ask whether networks are "locally" efficient in a sense that neither links can be added nor severed to increase overall welfare, and if not, we want to know how overall welfare can be improved. We use the following two definitions in order to describe non-efficient networks.

Definition 4. *A network g is called* over-connected *(with respect to the welfare function w) if $\exists g' \subset g$ such that $w(g') > w(g)$.*

Definition 5. *A network g is called* under-connected *(with respect to the welfare function w) if $\exists g' \supset g$ such that $w(g') > w(g)$.*

[7] Therefore, all results that we derive for all pairwise stable networks, e.g. non-efficiency of pairwise stable networks, carry over to any stronger notion of stability.

[8] See Jackson and Wolinsky (1996) for a general statement and Jackson (2004) for some more examples.

A network is over-connected if it is "too dense" in the sense that overall welfare can be improved by cutting links. Similarly, under-connected networks are "not dense enough". Efficient networks are neither over-connected nor under-connected, while all supernetworks of efficient networks are either over-connected or efficient and all subnetworks are either under-connected or efficient. Note that for any given w, a network can satisfy both, one, or none of these two properties. To shed some light into the tension between stability and efficiency, we will ask whether and under what conditions stable networks are over-connected or not under-connected, respectively under-connected or not over-connected. Our approach is very general and can be applied to most models of network formation. We discuss the implications of our results for the tension between stability and efficiency and also some of the implications for the characterization of stability and efficiency. Furthermore, from the perspective of a social planner, this gives some insights whether to subsidize or to tax the formation of links in order to arrive in a socially preferred outcome.

3 Positive Externalities

We start by drawing our attention to network formation games with positive externalities. Positive externalities in network formation games simply capture that two players forming a link cannot decrease other players' utilities. Differently put, players experience positive effects on their utility from others forming a link. Defining it formally we get:

Definition 6. *A profile of utility functions u satisfies positive externalities if $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i, j\}$ it holds that*

$$u_k(g \cup ij) \geq u_k(g).$$

Being required for any network, any link and any player, this property seems quite restrictive. However, we argue that there are many such contexts, and we can easily find examples in the literature on strategic network formation that satisfy this property. Among them are "Provision of a pure public good" (Goyal and Joshi, 2006a), "Market sharing agreements" (Belleflamme and Bloch, 2004), and the "Connections model" (Jackson and Wolinsky, 1996), which we discuss below. In case of a utility function that is additive separable into costs and benefits (where costs only depend on own degree), positive externalities are implied by a simple monotonicity property of the benefit function. In this context, players have to carry the costs of their own links, but share the benefits with others. Intuitively, individual incentives to establish a link might be lower than its collective value because of positive externalities.

Thus a link could be socially desirable, but conflicts with the individual interest of both involved players. The other way around, however, is not possible: if two players agree to form a link, then this link is always socially desirable because of positive externalities. We need one additional property to strengthen this observation. Consider the following definition (taken from Hellmann (2009)):

Definition 7. *A profile of utility functions u is* **concave (in own links)***, if $\forall i \in N, \forall g \in G$, and $\forall l_i \subset L_i(g^N \setminus g), \forall ij \in g$ it holds*

$$mu_i(g, ij) \geq mu_i(g \cup l_i, ij).$$

Informally put, the property requires that the marginal contribution of a link for a player is decreasing in the set of links he already has. Hellmann (2009) shows that this concavity property is equivalent to two notions we find in the literature. The first one is called "convexity in own *current* links" (see e.g. Bloch and Jackson, 2007): A profile of utility functions u is convex in own current links if $\forall i \in N, \forall g \in G,$ and $\forall l \subseteq L_i(g)$ it holds that $mu_i(g,l) \geq \sum_{ij \in l} mu_i(g, ij)$. The second property equivalent to concavity is known as "(1-)concavity in own *new* links" (see Calvó-Armengol and Ilkiliç, 2007): A profile of utility functions u is concave in own new links if for all $i \in N$, for all $g \in G$ and for all links l such that $l \subseteq L_i(g^N)$, and $l \cap g = \emptyset$ the following holds: $mu_i(g \cup l, l) \geq \sum_{ij \in l} mu_i(g \cup ij, ij)$. We will use this equivalence in the proofs of Theorem 1, Theorem 2, and Theorem 3.

With these definitions in hand, we can now formalize the intuition that stable networks are rather *not dense enough* compared to efficient networks. In fact, the following result shows that a stable network can never be improved by the deletion of links.

Theorem 1. *If a profile of utility functions u satisfies positive externalities and concavity, then no pairwise stable network is over-connected with respect to any monotonic welfare function w, that is $\forall g \in [PS(u)]$ it holds that $\nexists g' \subset g : w(g') > w(g)$.*

All proofs can be found in the appendix. To prove this result, we show that any player is worse off in a subnetwork g' of a PS network g.[9] Pairwise stability together with concavity implies that a player cannot prefer a network $\tilde{g}(\subset g)$ that has only been reduced by some of his own links. Because of positive externalities, he cannot prefer a subnetwork $g' \subset \tilde{g}$ of the reduced network.

Positive externalities are one source for the inefficient outcome of under-connected networks. A second source can be the miscoordination between two players. Since pairwise stability assumes that each link needs bilateral consent, it can happen that one player i cuts a link (ij) although j would have heavily benefited from that link. Pairwise stability with transfers excludes this source of inefficiency. While PS requires that no agent improves his utility by cutting a link, pairwise stability with transfers is a bit weaker in this respect (because the other player involved can compensate him for keeping the link). To establish the corresponding result for pairwise stability with transfers, we restrict attention to the utilitarian welfare function.

Theorem 2. *If a profile of utility functions u satisfies positive externalities and concavity, then no $g \in [PS^t(u)]$ is over-connected with respect to the utilitarian welfare function.*

The assumptions on the utility function, namely positive externalities and concavity, appear in many models of networks formation, some of which will be analyzed subsequently. There are models however that only fulfill the assumptions of the theorems for a smaller domain of networks.[10] For two networks $g, g' \in G$ such that $g \subset g'$, let the set $[g, g']$ be defined as the set of all networks containing g and being contained in g', $[g, g'] := \{g'' \in G | g \subseteq g'' \subseteq g'\}$. When requiring the assumptions to hold only for a smaller domain $\tilde{G} \subset G$, with the additional assumption that $[g^{\emptyset}, g] \subseteq \tilde{G}$ for all $g \in \tilde{G}$, the results carry over to that domain.

[9] The result also implies that no subnetwork of a PS network is Pareto better, but is much stronger than that.

[10] In the case of negative externalities this is shown in the example of free trade agreements.

Remark 1. *Suppose u satisfies positive externalities and concavity in own links on a domain $\tilde{G} \subseteq G$, such that for all $g \in \tilde{G}$ it holds that $[g^{\emptyset}, g] \subseteq \tilde{G}$. Then no $g \in \tilde{G}$, which is pairwise stable with transfers or pairwise stable is over-connected with respect to the utilitarian welfare function.*

The results excluding over-connectedness have trivial implications for the complete and empty network. As any network is a subnetwork of the complete network, it follows that (a) if the complete network is stable, then it must also be efficient. Since any network is a supernetwork of the empty network it follows that, (b) if the empty network is uniquely efficient, then no other network can be stable. Next, we study some models of network formation from the literature and show how to apply our results.

The Connections Model Revisited

The connections model was introduced in Jackson and Wolinsky (1996). It models the flow of resources (like information or support) via shortest paths in a network. Let $d_{ij}(g)$ denote the distance of players i and j in network g (which is defined to be ∞ for unconnected pairs), then the utility of each player can be written as

$$u_i^{CO}(g) = w_{ii} + \sum_{j \neq i} \delta^{d_{ij}(g)} w_{ij} - \sum_{j:ij \in g} c_{ij}, \quad \textit{with} \;\; \delta \in (0,1). \tag{1}$$

It is easy to see that the connections model satisfies positive externalities. If ij forms in some network g, then the utility of a player $k \neq \{i,j\}$ either does not change or increases as some of k's distances are shortened because $d_{km}(g \cup ij) \leq d_{km}(g)$ for all ij and m. Moreover, it can be shown that that $u^{CO}(\cdot)$ satisfies concavity. By the result of Hellmann (2009) it suffices to show that u^{CO} satisfies convexity in own current links, that is $\forall i \in N, \forall g \in G,$ and $\forall l \subseteq L_i(g)$, it holds that $mu_i(g,l) \geq \sum_{ij \in l} mu_i(g,ij)$. This has been done by Calvó-Armengol and Ilkiliç (2007) for the symmetric connections model. We make a straightforward generalization of their proof.[11]

Lemma 1. *The heterogeneous connections model satisfies concavity.*

Consequently (by Theorem 1 and Theorem 2), no pairwise (Nash) stable network can be over-connected w.r.t any monotonic welfare function and no pairwise stable network with transfers can be over-connected w.r.t. the utilitarian welfare function. While stable networks depend on the dyadic specifications of value and costs (w_{ij}, c_{ij}), the results excluding over-connectedness imply that the welfare of a stable network can never be improved by severing links.

There are more specific results for the connections model in its symmetric version, setting $w_{ij} = 1$, $c_{ij} = c$ ($\forall i \neq j$) and considering the utilitarian welfare function w^u only. This has been studied in Jackson and Wolinsky (1996), Jackson (2003), Hummon (2000), and Buechel (2008) among others. Jackson and Wolinsky (1996, Prop. 1 and Prop. 2) show

[11] In fact, such a generalization can be made for any distance-based utility function in the sense that benefits are decreasing with distances and costs only depend on direct links.

that for low costs ($c < \delta - \delta^2$) the complete network is efficient (and uniquely pairwise stable); for medium costs ($\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$) the star network is efficient; while for very high costs ($c > \delta + \frac{n-2}{2}\delta^2$) the empty network is efficient. Their famous statement of inefficiency in the connections model is the following: "For $\delta < c$, any pairwise stable network which is non-empty is such that each player has at least two links and thus is inefficient."[12]

What does our result excluding over-connectedness add to their discussion of inefficiency? First, there is the above mentioned trivial implication for the empty network: Since any network is a supernetwork of the empty network, it follows that if the empty network is uniquely efficient, then no other network can be stable. Thus, the statement of inefficiency is restricted to $\delta < c < \delta + \frac{n-2}{2}\delta^2$. Second, the result on over-connectedness adds a new point of view on the flavor of inefficiency. This can be illustrated in the following example, which is also taken from Jackson and Wolinsky (1996, Ex. 1).

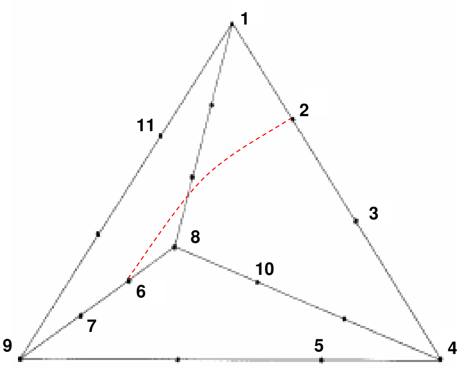

Figure 1: Example of an inefficient network ("Tetrahedron").

Example 1. *The network in fig. 1, called "Tetrahedron", is stable for costs $c > \delta$, where the star network is uniquely efficient.*[13] *The tetrahedron is "too dense" in the sense that it has* 18 *links, while the efficient network has* 15*. Accordingly, Jackson and Wolinsky (1996, p. 51) label it as "over-connected". However by Theorem 1, it is not over-connected according to the definition used in this paper. This means that the welfare of the tetrahedron cannot be improved by leaving out some of its links. Moreover, we claim that the tetrahedron is under-connected in the parameter range such that it is pairwise stable. In the appendix we show that the addition of a link between the players "2" and "6" would strictly improve utilitarian welfare. The same point as in the Tetrahedron can be illustrated in a circle graph of $n \geq 7$: both networks are under-connected for any costs for which they are pairwise stable.*

The example illustrates two different viewpoints on inefficiency (in the connections model). From the viewpoint of a social planer that can unrestrictedly manipulate a given network, some stable networks are "too dense" in the sense that less links are needed to form the efficient one. From the viewpoint of a social planer who is restricted to either foster or

[12] Jackson and Wolinsky (1996), p. 51.

[13] More precisely, g^{Tetra} is pairwise stable iff $\delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4) \leq c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5)$.

hinder the formation of links (e.g. by taxes or subsidies), many stable networks in the connections model are “not dense enough” (under-connected), while none is “too dense” (over-connected).

Market Sharing Agreements

Besides the connections model, it is easy to find further examples for positive externalities (and concavity in own links). Among them is the model of “market sharing agreements” described in Goyal and Joshi (2006a). In this model, there are n firms and n markets, where each firm has one home market and can be active in all other markets, too. Before starting a Cournot competition in each market, bilateral agreements can be made to stay out of each other’s home market.

The reduction of competitors in the own market might be profitable. However, all remaining competitors in the market benefit from these activities without paying for it. That is why the utility function of this example exhibits positive externalities. In addition, it satisfies concavity, such that both results above (Theorem 1 and Theorem 2) apply. Consequently, positive externalities lead to rather too few agreements with respect to a monotonic welfare function. Note that such a function only covers firms’ utility, but not consumers’.

Provision of a Pure Public Good

Another example of a network formation model that satisfies the assumptions of Theorems 1 and 2 is the “provision of a pure public good” model by Goyal and Joshi (2006a) who extended a model of Bloch (1997). In this model, n players choose an output level x_i (second stage), which is valuable for everybody $\tilde{\pi}_i(x) = \sum_{i \in N} x_i$. Collaboration (knowledge sharing) between any two players is costly, but can reduce the marginal costs of producing the output (first stage).[14] Assuming that any player chooses his output quantity optimally, the utility of a player i is:

$$u_i^{PG}(g) = \frac{1}{2}(d_i(g)+1)^2 + \sum_{j \in N \setminus i} (d_j(g)+1)^2 - cd_i(g),$$

where the first term is the difference of own output and production costs, the second term is the output of all other agents, and the last term is the costs of collaboration.

Not surprisingly, the network formation situation of the first stage satisfies positive externalities because other agents’ cooperations lower their costs, increase their optimal

[14] Agent i’s cost of producing the output is $f_i(x_i, g) = \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2$. Fixing the number of collaborators $d_i(g)$, the utility maximizing output quantity of an agent i can be derived by $\max_{x_i \in \mathbb{R}_+} x_i + \sum_{j \in N \setminus i} x_j - \frac{1}{2}(\frac{x_i}{d_i(g)+1})^2 =: F(x)$. This yields $F'(x) = 0 \iff x_i^* = (d_i(g)+1)^2$. Then, plugging in the optimal output $(F(x^*))$ for any agent into the objective function and subtracting the linking costs yields the utility of one agent.

output and, hence, is beneficial to all. To see this, observe that the addition of foreign links increases the middle term of the utility function. Note that in this example the externalities are strict in the sense that the addition of any link in any network increases the utility of all agents that are not involved. Moreover, given these specific functional forms, u satisfies concavity. Thus, the two results (Theorem 1 and Theorem 2) imply again that no stable network can be over-connected.

Consider very low costs $c \leq \frac{3}{2}n^2 - \frac{3}{2}(n-1)^2 =: lb$ such that the complete network is stable. By the results (Theorem 1 and Theorem 2) above, the complete network must also be efficient for these costs. In fact, since the externalities are strict, there exists an $\epsilon > 0$ such that g^N is efficient for $c \leq lb + \epsilon$. The tension can be illustrated for $lb < c < lb + \epsilon$. In this cost range the complete network would still be efficient. However, the stable networks are not complete.

The model can be interpreted as a doubled public goods problem. In the second stage there is the classic public goods problem, where individual output x_i is chosen "too low" (from a collective perspective). This problem persists, but in addition (in the first stage) players tend to choose "too few" links reducing the cost of provision, such that the outcome is even worse. In the same manner any network formation situation with positive externalities can be interpreted as a public goods problem. Utility maximizing agents simply do not internalize the positive effects that establishing a bilateral link means for other agents.

4 Negative Externalities

Negative externalities in network formation occur, when any addition of a link cannot be beneficial for the players which are not involved in this link. Formally, we speak of weakly negative externalities (further denoted as negative externalities) if the following holds:

Definition 8. *A profile of utility functions u satisfies negative externalities if $\forall g \in G, \forall ij \notin g, \forall k \in N \setminus \{i,j\}$ it holds that*

$$u_k(g \cup ij) \leq u_k(g).$$

When considering negative externalities in economics, equilibrium analysis usually shows that individuals do rather "too much" (pollute, etc.) than being socially optimal. In that sense it is intuitive to think about "too dense" networks as being stable. For stability only individual incentives are considered and not the overall welfare. Network formation games with negative externalities are thus expected to be over-connected, and not under-connected. While the intuition points into the direction of rather "too dense" networks as being stable, the bilateral nature of link formation does not necessarily exclude pairwise stable networks from being under-connected. As discussed above any player has the power to deny links.[15] Although the concept of pairwise stability already solves some miscoordination problems that arise in the bilateral network formation of the Myerson game, it might happen that a player rejects a link that is highly beneficial to the proposing

[15]In section 5 when introducing a general framework that is not restricted to bilateral and unweighted network formation we treat this as a property of the outcome function, which we call *rejection power*.

player. This link, however, could potentially lead to higher welfare, which we aim to exclude.

A simple way out of this issue is using the stability concept "Pairwise Stability with transfers" $[PS^t]$. This concept helps ensure that any single link that is not in a network g, which is pairwise stable with transfers, cannot be welfare improving. Analogously to our results on positive externalities, we require concavity (see Definition 7) for our main result on negative externalities to ensure that no set of links can be welfare improving. Utility functions satisfying this property imply that networks, which are pairwise stable with transfers are not under-connected with respect to the utilitarian welfare function:

Theorem 3. *Suppose a profile of utility functions u satisfies negative externalities and concavity, then no network $g \in G$, which is pairwise stable with transfers, is under-connected with respect to the utilitarian welfare function.*

As usual the proof can be found in the appendix. Pairwise stability with transfers differs from pairwise stability significantly. In general, neither $[PS(u)] \subseteq [PS^t(u)]$ nor $[PS(u)] \supseteq [PS^t(u)]$. The concept of pairwise stability with transfers rather stems from a game, in which players can pay transfers to make others willing to build a link. Formally, pairwise stability with transfers has been introduced in Bloch and Jackson (2007). Although the concepts differ, we can easily find properties of the utility function that are sufficient to ensure equivalence of both stability concepts. Intuitively, both concepts should coincide if for any link the gains or losses for both involved individuals coincide. This is the case in network formation games that have potential function.[16] The property that we need in order to show equivalence of the concepts, however, only requires individuals to either both gain or lose from a link. The following definition taken from Chakrabarti and Gilles (2007) formalizes this intuition:

Definition 9. *A profile of utility functions u satisfies pairwise sign compatibility (PSC), if for all $g \in G$ and for all links $ij \in g$ it holds, that:*

$$sgn\left(mu_i(g, ij)\right) = sgn\left(mu_j(g, ij)\right)$$

Pairwise sign compatibility requires any two players to either both gain, both lose or both be indifferent from a mutual link in any network. Under this condition the power to deny any link does not play a role, since any rational denial of a link by one player implies that the link is not beneficial for both involved players. In particular no transfer of utility would make sense in a network, that is already pairwise stable, and conversely if no transfer for any link is beneficial, then no addition or deletion of a link is beneficial for either player, meaning that the network is pairwise stable. Thus, both stability concepts coincide and we get:

Theorem 4. *Suppose that a profile of utility functions satisfies PSC. Then,*

$$[PS(u)] = [PS^t(u)].$$

As a consequence of Theorem 3 and Theorem 4 it is obvious that under PSC, concavity and negative externalities, no pairwise stable network can be under-connected.

[16] For a study of network potentials, see Chakrabarti and Gilles (2007).

Corollary 1. *Suppose that a profile of utility functions satisfies negative externalities, PSC and concavity in own links. Then, no network g which is pairwise stable is under-connected with respect to the utilitarian welfare function.*

The proof is an immediate implication of Theorem 3 and Theorem 4 and thus is omitted. For both results we can restrict the domain of networks for which the assumptions have to hold resembling Remark 1.

Remark 2. *Suppose u satisfies negative externalities and concavity on a domain $\tilde{G} \subseteq G$, such that for all $g \in \tilde{G}$ it holds that $[g, g^N] \subseteq \tilde{G}$. Then no $g \in \tilde{G}$, which is pairwise stable with transfers is under-connected with respect to the utilitarian welfare function. If u additionally satisfies PSC on $\tilde{g}$, then no pairwise stable network is under-connected with respect to the utilitarian welfare function.*

The remark is shown in the appendix. It becomes very handy when analyzing network formation games that fulfill the properties of Theorem 3 only on a certain domain $\tilde{G}$. We will see examples of network formation games such that the utility function is concave only for networks such that all players have at least one link,[17] and it is straightforward to see that for such a set of networks $\bar{G} := \{g \in G | d_i(g) \geq 1 \; \forall i \in N\}$ it holds that $g \in \bar{G} \Rightarrow [g, g^N] \subseteq \bar{G}$. Another example of a domain of networks for which the above property is satisfied is the set of connected networks.[18]

We require pairwise stability with transfers or the pairwise sign compatibility for our result, since it is a way to limit the gain of one of the two players from a potential mutual link. Without assuming pairwise sign compatibility or transfers, we need to compare the sum of utilities of non-involved players to the sum of utilities of involved players for any new link to exclude large gains and thus stable networks from being under-connected. If we cannot exclude possible gains from additional links in pairwise stable networks, we may, however, ask whether a pairwise stable network is not Pareto under-connected, that is whether there does not exists a Pareto better supernetwork.

Definition 10. *A network $g \in G$ is Pareto under-connected if there exists a network $g' \supset g$, such that $u_i(g') \geq u_i(g)$ for all $i \in N$, and there exists a $j \in N$ such that $u_j(g') > u_j(g)$.*

To exclude Pareto under-connectedness, the following definition of transitivity is needed:

Definition 11. *A utility function satisfies transitivity of negative (positive) marginal utility in new links for $g \in G$ if for all ij, $jk \notin g$ the following holds:*

$$mu_i(g, ij) < (>)0, \;\; mu_j(g, jk) < (>)0 \quad \Rightarrow \quad mu_i(g, ik) < (>)0.$$

This definition captures the idea of transitivity: if player A does not want to connect to B and B does not want to connect to C, then A does not want to connect to C. The following Theorem presents the result that no pairwise stable network can be Pareto under-connected.

[17] The model of free-trade agreements satisfies concavity only for networks such each players degree is at least one.

[18] The set of connected networks is formally defined as $G^{con} := \{g \in G \mid \;\; \forall i, j \in N, \;\; d_{ij}(g) < \infty\}$.

Theorem 5. *Suppose a profile of utility functions satisfies negative externalities, and concavity. If g is pairwise stable and u satisfies transitivity of negative marginal utility for g, then g cannot be Pareto under-connected.*

In the proof of the result, we need the transitivity property in order to exclude the case that a set of links l can be added to a pairwise stable network such that each involved individual has at least two links in l. This set of links can be potentially beneficial, if for each player one link is highly beneficial (and thus a loss for the other players) and compensates for the loss of other links (which could mean gains for the other players, which are involved). The property of transitivity, however, is only needed locally, that is it is only needed for pairwise stable networks. It could thus also be seen as a stability refinement.

Degree Dependent Utility Functions

A lot of applications and examples of network formation games with negative externalities can be found under what we call "degree dependent utility functions". For instance, Goyal and Joshi (2006a) present network formation models, where the utility functions depend only on the number of own links, the number of links of neighbors and on the number of links of non-neighbors. They analyze two different models:

- Playing the field:
$$u_i^{PF}(g) = \Phi\big(d_i(g), D(g_{-i})\big) - d_i(g)c. \tag{2}$$
- Local Spillovers:
$$u_i^{LS}(g) = \Psi_1(d_i(g)) + \sum_{j \in N_i} \Psi_2(d_j(g)) + \sum_{k \notin N_i} \Psi_3(d_k(g)). \tag{3}$$

Here g_{-i} represents the network, obtained by deleting player i and all his links and $D(g) = \sum_{i \in N} d_i(g)$. The first utility function, playing the field, presented by Goyal and Joshi (2006a) features a very specific structure in that each utility does not depend on network positions, but rather on the number of own and the sum of all other players' links. In contrast, the local spillover utility function includes a little bit more of the network structure. Here, neighborhood distribution of links matters. In the playing the field case, players do not care to whom to connect, they are not able to distinguish between different players, whereas in the local spillover case, players are able to distinguish between players and might have preferences of whom to connect to. For the following we will generalize both cases and try to shed some light on the tension between stability and efficiency when considering the following utility function:

Definition 12. *A profile of utility functions is called homogeneous degree dependent utility function if for all $i \in N$ there exist function $f : \mathbb{R}^n \to \mathbb{R}$ such that:*

$$u_i^{DD}(g) := f\left(d_i(g), \big(d_j(g)\big)_{j \in N_i(g)}, \big(d_k(g)\big)_{k \notin N_i(g)}\right). \tag{4}$$

This utility function captures both the local spillovers function and the playing the field function, as well as other examples. The distinction between neighbors and non-neighbors is only made for illustration since in a lot of examples those degrees are treated differently. Abusing notation, we will also write $u_i^{DD}\big(d_i, (d_j)_{j\in N_i}, (d_k)_{k\notin N_i}\big)$ instead of $u_i^{DD}(g)$.

For the analysis of both playing the field and general degree dependent utility functions, we focus on the case of negative externalities, since the tension between efficiency and pairwise stability seems to be sufficiently covered in section 3. In contrast the case of negative externalities requires the properties transfers or pairwise sign compatibility in order to show the result.

In the playing the field case, negative externalities imply that u^{PF} is decreasing in its second argument, i.e. $u^{PF}(l, k+1) - u^{PF}(l, k) \leq 0$ for all $l = \{0, ..., n-1\}$, $k = \{0, ..., n-2\}$. General degree dependent utility functions satisfy negative externalities if for all $i \in N$, for all $d_i \in \{0, ..., n-1\}$, and for all $d_{-i}, \tilde{d}_{-i} \in \{0, ..., n-1\}^{n-1}$ such that $\tilde{d}_k \geq d_k$ for all $k \in N \setminus \{i\}$, the following holds:

$$u_i^{DD}(d_i, d_{-i}) \geq u_i^{DD}(d_i, \tilde{d}_{-i}).$$

For the special cases of degree dependent utility functions, the results of section 4 carry over and some assumptions are automatically satisfied. For playing the field utility functions, it becomes immediately clear from (2), that u^{PF} satisfies our notion of transitivity for all $g \in G$ because either a player wants a link to any remaining player or to none. Thus the transitivity condition of Theorem 5 is satisfied, and we get the following corollary :

Corollary 2. *Suppose that the profile of utility functions of a network formation game is given by (2), and u^{PF} satisfies negative externalities and concavity. Then no pairwise stable network g is Pareto under-connected.*

Due to the special structure and homogeneity of the utility functions given by (2) and (3), it is often observed that regular networks are pairwise stable, i.e. feature equal degree distributions.[19] Since the utility functions given by (4) are homogeneous, regular pairwise stable networks are also pairwise stable with transfers. This is true since no single player wants to add or cut a link in regular pairwise stable networks. Furthermore, the star is a common observed stable network. In the star, $n-1$ players share equal degree, and the other player is completely connected, and cannot add any links. Thus, the same considerations hold for the star. These observations lead to the following result:

Corollary 3. *Suppose that the profile of utility functions is given by (4) and, furthermore, satisfies negative externalities and concavity. Then, no regular pairwise stable network is under-connected. Moreover if the star is pairwise stable, then it is not under-connected.*

The corollary also holds for playing the field and local spillover utility functions since degree based utility functions satisfying (4) contain these two cases.

Applications of degree dependent utility functions can be found plenty. Among those are the provision of a public good and the market sharing agreements presented in section 3,

[19] See Goyal and Joshi (2006a) for a detailed analysis of stable networks in playing the field and local spillover games.

but also there are several examples for negative externalities, of which we present some below.

The Co-author Model

The co-author model has been introduced by Jackson and Wolinsky (1996) in their seminal paper and describes the utility of joint work. The nodes of the network are interpreted as researchers, who spend time writing papers. A link between two researchers i and j represents a collaboration between both researchers. The amount of time a researcher spends on a project is inversely related to the number of projects he is involved in. The payoff function is given by:

$$u_i^{CA}(g) = \sum_{j \in N_i(g)} \left(\frac{1}{d_i(g)} + \frac{1}{d_j(g)} + \frac{1}{d_i(g) d_j(g)} \right) = 1 + \left(1 + \frac{1}{d_i(g)}\right) \sum_{k \in N_i(g)} \frac{1}{d_j(g)},$$

and $u_i^{CA}(g) = 0$ if $d_i = 0$. The utility depends only on own degree and neighbors degree and thus is degree dependent. Obviously, the functional form satisfies negative externalities as the utility of players decrease, when neighbors are adding links, i.e. increasing their degree. From Jackson and Wolinsky (1996) we get that if n is even, then any efficient network consists of $n/2$ separate pairs. Any pairwise stable network, however, can be partitioned into fully intra-connected components, each of which has a different number of members (if m is the number of members of one such component and k is the next largest size, then $m > k^2$).

We can add to this result that none of the stable networks contains a singleton component since each player is better off connecting to some player than to none, and each player i wants a link to a player j, for whom $d_j \leq d_i$. For n being even it is clear, that any pairwise stable network contains an efficient network, since any network consisting of $n/2$ separate pairs is efficient. For n being odd, any network that consists of $(n-2)/2$ separate pairs and three players, which are connected by 2 links, is efficient. We can show that also in the case of n being odd that any pairwise stable network contains an efficient network, implying over-connected pairwise stable network.

Proposition 1. *In the co-author model if $n \geq 3$, then any pairwise stable network is over-connected.*

The proof is straightforward and uses the result of Jackson and Wolinsky (1996) component-wise (for any completely connected component of the pairwise stable networks). It is welfare better for any component of at least size three to be connected like one of the efficient networks. Thus any component of any pairwise stable network contains a welfare better subcomponent, implying that any pairwise stable network contains a welfare better subnetwork.

Patent Races

Goyal and Joshi (2006a) derive this model as a variation of the classical patent race model.[20] In addition to the classical model, firms can join R&D collaborations to accelerate research. The first firm to develop the new product is awarded a patent. The random time $\tau(l_i(g))$ at which the innovation happens is given by

$$Pr(\{\tau(d_i(g)) \leq t\}) = 1 - \exp(-d_i(g)t).$$

Assuming risk neutrality, payoff of 1 in case of receiving the patent and 0 else, and a discount factor ρ, a firm i gets the following expected payoff:

$$\begin{aligned} u_i^{PR}\big(d_i(g), D(g_{-i})\big) &= E_t[exp(-\rho t)Pr(\tau(d_i(g)) = t)\prod_{j \neq i} Pr(\tau(d_j(g)) > t)] - d_i(g)c \\ &= \frac{d_i(g)}{\rho + D(g)} - d_i(g)c \;=\; \frac{d_i(g)}{\rho + 2d_i(g) + D(g_{-i})} - d_i(g)c. \end{aligned}$$

This model is thus a playing the field utility function. Moreover, it satisfies negative externalities since links of other firms reduce the probability to innovate firstly. Also, since u_i^{PR} is a concave function of $d_i(g)$, it is concave according to Definition 7. From Theorem 3 we can thus conclude that no pairwise stable network with transfers is underconnected. In fact, it is straightforward to calculate the efficient networks since the utilitarian welfare is given by:

$$w^{PR}(g) = \sum_{i \in N} u_i^{PR}(g) = \sum_{i \in N} \left(\frac{d_i(g)}{\rho + D(g)} - d_i(g)c \right) = \frac{D(g)}{\rho + D(g)} - D(g)c.$$

In this case the utilitarian welfare only depends on the total number of links and thus any network that contains the optimal number of total links is efficient. The distribution of links and the structure of the network do not matter for efficiency. We can easily calculate that for $\frac{\rho}{(\rho+2(k+1))(\rho+2k)} < c < \frac{\rho}{(\rho+2k)(\rho+2(k-1))}$ any network which contains k links is efficient and no other networks are efficient.

It requires a little bit more to characterize stable networks. However, for this matter we can apply Theorem 3 in order to bound the total number of links.

Proposition 2. *Suppose that* $\frac{\rho}{(\rho+2k+2)(\rho+2k)} < c$, *then all networks* g, *which are pairwise stable with transfers, have to contain more than* k *total links, in other words* $D(g) \geq 2k$.

In their paper, Goyal and Joshi (2006a) only find a partial characterization for the set of pairwise stable networks. By applying Theorem 3 we were able to contribute to their characterization. This example shows that Theorem 3 not only describes the tension between stability and efficiency, but it can also be applied to characterize the stable networks (resp. the efficient ones).

[20] See Dasgupta and Stiglitz (1980) among others.

Free Trade Agreements

This model has been introduced by Goyal and Joshi (2006b). We analyze the most basic setup here. In this example there are n countries. In each country there is one firm producing a homogeneous good. The firm may sell the product in the domestic market as well as in foreign markets. If two countries do not have a free trade agreement (FTA) the importing country charges tariffs. Given a configuration of FTA's the firms then compete in each market by choosing quantities. We denote the quantity output of firm j in country i by Q_i^j. In each country $i \in N$, a firm faces an identical inverse demand given by $P_i = \alpha - Q_i$, where $Q_i = \sum_{j \in N} Q_i^j$, and $\alpha > 0$. All firms have a constant and identical marginal cost of production, $\alpha > \gamma > 0$. In the basic model, linear demand and identical tariffs are assumed. Forming an FTA lowers the tariff to 0. Assuming high tariffs $T > \alpha$, firm i sells in country j if and only if there is a FTA between the two countries, i.e. $ij \in g$. Given firm i is active in market j, then its output is given by $Q_j^i = \frac{\alpha-\gamma}{d_j(g)+1}$. The utility function of country i is given by its "social welfare" derived as the sum of consumer surplus, firm's profits and tariff revenue, which can be simplified in our basic setup:

$$
\begin{aligned}
u_i^{FTA}(g) &= 1/2 Q_i^2 + \Big((P_i(g) - \gamma) Q_i^i(g) + \sum_{j \neq i} (P_j(g) - \gamma - T_j^i(g)) Q_j^i(g) \Big) + \sum_{j \neq i} T_i^j(g) Q_i^j(g) \\
&= 1/2 \left(\frac{(\alpha - \gamma)(d_i(g)+1)}{d_i(g)+2} \right)^2 + \sum_{j \in (N_i(g) \cup \{i\})} \left(\frac{\alpha - \gamma}{d_j(g)+2} \right)^2 .
\end{aligned}
$$

Again, this utility function is degree dependent. Moreover, it is straightforward to see that u^{FTA} satisfies negative externalities. To see the reasoning for negative externalities, suppose a free trade partner j of country i signs a free trade agreement with country k. Then firm k will enter market j and thus reduces the Cournot output of firm i, lowering the country's welfare function. If two non-trade partners of i sign a free trade agreement, then i's payoff remains unaffected. Straightforward calculations show that u_i^{FTA} satisfies concavity whenever $d_i \geq 1$. From Goyal and Joshi (2006b) we get the following characterization of stability and efficiency:

Proposition 3. (Goyal and Joshi, 2006b) *The complete network is a stable trading network. Furthermore, a network such that one component has $n-1$ countries and is complete and the other component is a single country can be stable. However, the complete network is uniquely efficient.*

In this case, the network such that there is one component with $n-1$ countries and one unconnected country is under-connected. It can be shown, that this network is also pairwise stable with transfers if n is sufficiently large. These networks however, do not fall in the restricted domain in Remark 2 since concavity only holds for u_i^{FTA} if $d_i \geq 1$. Thus Remark 2 only applies to all $g \in \tilde{G}$ such that $\tilde{G} := \{g \in G \mid \ d_i(g) \geq 1 \ \ \forall i \in N\}$, and hence does not exclude the case of under-connected pairwise stable networks $g \notin \tilde{G}$.

5 Extensions

So far, we have only considered undirected and unweighted networks. In this section we relax those assumptions by providing a general framework for the formation of networks and generalizing our main results. Particularly, our framework captures models of directed network formation including those in Bala and Goyal (2000) and models of weighted network formation of Bloch and Dutta (2005) and Rogers (2006). We show in this very general setup that we are able to reestablish the results discussed above. Moreover, by considering Nash stable networks, the statements get stronger.

5.1 Framework

As before, let N be a set of agents. In the following we define a network g to be a $n \times n$-matrix $[g_{ij}]_{i,j \in N}$, where an entry $g_{ij} \in \mathbb{R}$ is the weight of the relation from agent i to agent j. In the context of *directed* networks g_{ij} stands for the intensity/strength of the link $\vec{ij}$ which is different from the link $\vec{ji}$. In the context of *undirected* networks $g_{ij} = g_{ji}$ is interpreted as the intensity of the link ij. Unweighted networks are represented by $g_{ij} \in \{0, 1\}$ for all $i, j \in N$, where the intensity 0 or 1 indicates whether a link is present. A loop (or self-link) $g_{ii} \neq 0$ may stand for the resources (e.g. time, money) devoted to oneself. A negative tie $g_{ij} < 0$ incorporates a negative relation, such as being enemies.

Let $\mathcal{G} := \{[g_{ij}]_{i,j \in N} | g_{ij} \in \mathbb{R}\}$ be the set of all $n \times n$-matrices. We will restrict this domain according to the application in mind. For example G, the set of networks we considered in the sections above, can be written as a subset of $\mathcal{G}$.[21] Each player has a preference ordering over the set of networks, which we represent by a utility function $\mathfrak{u}_i : \mathcal{G} \to \mathbb{R}$. Let us now generalize Myerson's link announcement game.

A Generalized Link Announcement Game

Consider the following game form (N, S, Ψ) where N is the set of players as above, $S_i := \mathbb{R}^n$ is player i's strategy set and $\Psi : S \to \mathcal{G}$ is an outcome rule mapping strategy profiles into networks. The strategy set of each player is interpreted as intensities that a player can announce for the relationship with another player. We will also sometimes refer to strategies as an effort that player i invests into the relationship with j. The outcome rule Ψ defines how individual strategies translate into network links. In virtually any model, the intensity g_{ij} is only dependent on s_{ij} and s_{ji} and it does not decrease, ceteris paribus, with an increase in one of those. Therefore, we will assume for the following that Ψ is fully described by a function ψ that translates individual efforts into link strengths, $g_{ij} = \psi(s_{ij}, s_{ji})$, where ψ is non-decreasing function in both arguments.[22]

Consider the following examples of outcome rules, which lead to a reformulation of well-

[21]In particular, $G = \{g \in \mathcal{G} : \; g_{ii} = 0, \forall i \in N; \; g_{ij} = g_{ji} \in \{0, 1\}, \forall i, j \in N\}$ is the set of undirected unweighted networks without loops.

[22]This assumption restricts outcome rules such that the intensity g_{ij} cannot be determined by the strategy of some player $k \notin \{i, j\}$. Of course, that does not hinder agents from interacting strategically.

known models from the literature:

I) $g_{ij} = \psi(s_{ij}, s_{ji}) = \begin{cases} 1 & \text{if } i \neq j \text{ and } \min\{s_{ij}, s_{ji}\} \geq 1 \\ 0 & \text{else} \end{cases}$. Since link strength is either 0 or 1, this models the formation of unweighted networks. Moreover, networks are undirected because $\psi(s_{ij}, s_{ji}) = \psi(s_{ji}, s_{ij})$. Finally, the networks may not exhibit loops because $g_{ii} = 0 \ \forall i \in N$. Thus, we are in the context of unweighted undirected networks without loops. In fact, this is a reformulation of Myerson's link announcement game, also called the *Consent Game*. Strategies in this model are simply interpreted as players announcing the links they desire, i.e. $s_{ij} \geq 1$, while links are only formed if the two involved players agree about it.

II) $g_{ij} = \psi(s_{ij}, s_{ji}) = \phi(s_{ij}) + \phi(s_{ji}) \in [0, 1]$. This outcome rule allows to form undirected networks with link intensities in the interval $[0, 1]$. Individual efforts are perfect substitutes. Such a model for the formation of undirected weighted networks can be found in Bloch and Dutta (2005) (see "Assumption 1").

III) $g_{ij} = \psi(s_{ij}, s_{ji}) = \phi(\min\{s_{ij}, s_{ji}\}) \in [0, 1]$. This outcome rule captures the second model of Bloch and Dutta (2005) (see "Assumption 2"). It models the formation of undirected weighted networks, where individual efforts are perfect complements.

IV) $g_{ij} = \psi(s_{ij}, s_{ji}) = \begin{cases} 1 & \text{if } i \neq j \text{ and } s_{ij} \geq 1 \\ 0 & \text{else} \end{cases}$. With this outcome rule link strength can only be 0 or 1, modeling the formation of an unweighted network. Moreover, g_{ij} need not be equal to g_{ji} such that the networks are directed. The models in Bala and Goyal (2000) belong to this class.[23]

V) $g_{ij} = \psi(s_{ij}, s_{ji}) = \phi(s_{ij}) \in \mathbb{R}_+$.[24] This outcome rule models the formation of undirected networks which are weighted. Here, the intensity of any link is determined by the strategy of a single player. Specifically, any player can determine his outgoing links. A model of this type can be found in Rogers (2006) (see "Model A").

VI) $g_{ij} = \psi(s_{ij}, s_{ji}) = \phi(s_{ji}) \in \mathbb{R}_+$. This outcome rule captures the second model of Rogers (2006) (see "Model G"). It models the formation of directed weighted networks, where any player can determine his incoming links.

As the examples show, a choice of outcome rule determines the set of feasible networks in a model, $G^\Psi := \{g \in \mathcal{G} | \exists s \in S : \Psi(s) = g\}$. This completes the generalized society $(N, G^\Psi, \mathfrak{u})$. The six examples of outcome rules incorporate models of weighted and unweighted, directed and undirected networks, but do not cover loops or negative links. Such models, although reasonable, can hardly be found in the literature. To complete the definition of a non-cooperative game $\Gamma = (N, S, \mathfrak{u}^\Psi)$, we assume that the payoffs $\mathfrak{u}^\Psi : S \to \mathbb{R}^n$ are defined by the network utility function function $\mathfrak{u}$ evaluating the induced network, $\mathfrak{u}^\Psi(s) := \mathfrak{u}(\Psi(s))$.

[23] Not only the one-way flow model, but also the two-way flow model is formalized as a directed network here because the utility of an agent does not only depend on the undirected links established, but also on who initiated them.

[24] Here ϕ may be specified according to the application in mind.

Now, we can make use of the well-established solution concepts for non-cooperative games. As usual, we define: s^* is a Nash Equilibrium in Γ if $\forall i \in N$, $\forall s_i \in S_i$ it holds that $\mathfrak{u}_i^\Psi(s_{-i}^*, s_i^*) \geq \mathfrak{u}_i^\Psi(s_{-i}^*, s_i)$. This leads to a basic notion of stability for a generalized society $(N, G^\Psi, \mathfrak{u})$.

Definition 13. *A network g is Nash stable (NS) if it can be supported by a Nash equilibrium in the corresponding link formation game Γ; formally, $g \in [NS(\mathfrak{u})]$ if and only if $\exists s \in S$ such that s is a Nash equilibrium in Γ and $\Psi(s) = g$.*

Several other notions of stability are reasonable. In particular, Nash stability is too weak for a reasonable stability concept in the Consent Game (outcome rule I) because, due to coordination problems, any network where no player wants to cut links is Nash stable. But, since the notion of Nash equilibrium is so weak, the results we present for Nash stable networks are strong as they hold for any refinement of Nash stability.

Note that the way we represent the examples of models above is not identical to their introduction in the literature. We embedded them into our framework without keeping the original notation and in the various models ϕ is further specified, e.g. to be concave, convex, satisfy $\phi(0) = 0$, etc. More importantly, we work with the strategy sets of the form $S_i = \mathbb{R}^N$ in any model. While this might lead to much more Nash equilibria in the space of strategy profiles, it does not lead to more Nash stable networks in the outcome space. For instance, in the original formulation of the Consent Game, strategy sets are $\{0, 1\}$, while in our formulation (outcome rule I) a player can announce any real number above 1 to initiate a link. However, the outcome rule is specified such that the set of networks G^Ψ is identical to the set of possible networks of the Consent Game in Myerson (1991), which we denoted by G in the former sections. Moreover, in the models of Rogers (2006) and Bloch and Dutta (2005) strategies are restricted. For example, Rogers (2006) introduces a budget constraint β_i for every agent such that $\sum_{j \in N} s_{ij} \leq \beta_i$, while $s_{ii} = 0$. Similar assumptions are made in Bloch and Dutta (2005). We will address this aspect below.

Some notation for a Generalized Situation of Network Formation

Consider a generalized society $(N, G^\Psi, \mathfrak{u})$. For G^Ψ, the set of feasible networks depending on the outcome rule Ψ, we can define similar operations as in Section 2. We call a matrix g' a subnetwork (supernetwork) of a network $g (\in G^\Psi)$, denoted as $g' \sqsubset (\sqsupset) g$, if it belongs to the set of feasible networks $g' \in G^\Psi$ and all entries in the matrix are smaller (larger) or equal, $g'_{ij} \leq (\geq) g_{ij}, \forall i, j \in N$. Similarly, we denote $g' \sqsubset_i g$ if g is a subnetwork of g' that differs only in links involving i, i.e. $g' \sqsubset_i g$ if and only if $g' \sqsubset g$ and $g'_{kl} < g_{kl}$ implies that $i \in \{k, l\}$.

In a given network, players may increase or decrease intensities of links. We use the following notation: For $g \in G^\Psi$ and $\delta \in \mathbb{R}$ we denote by $g +^{ij} \delta$ the matrix $[g'_{kl}]_{k,l \in N}$ with $g'_{kl} = g_{kl}$ $\forall (k, l) \neq (i, j)$ and $g'_{ij} = g_{ij} + \delta$, which means that the intensity of the link $\vec{ij}$ is increased by δ. This definition is only meaningful if the operation $g_{ij} + \delta$ is feasible, that is if $g' \in G^\Psi$.[25] In the context of undirected networks, each change of link strength

[25]For instance, Ψ has to present an outcome rule that allows for directed networks.

has consequences for two entries in the matrix which defines the network. Therefore, we define the following operation: For $g \in G^\Psi$ and $\delta \in \mathbb{R}$ we denote by $g \mp^{ij} \delta$ the matrix $[g'_{kl}]_{k,l\in N}$ with $g'_{kl} = g_{kl}\ \forall\{k,l\} \neq \{i,j\}$ and $g'_{ij} = g_{ij} + \delta$ and $g'_{ji} = g_{ji} + \delta$, which means that the intensity of link ij is increased by δ. Both operations coincide for changes of loop intensities, i.e. $g +^{ii} \delta = g \mp^{ii} \delta$.

The definitions of efficiency in Section 2 carry over from a society (N, G, u) to a generalized society $(N, G^\Psi, \mathfrak{u})$, only the domain has to be adapted. In brief, a welfare function $\mathfrak{w} : \mathbb{R}^n \to \mathbb{R}$ satisfies monotonicity if $\mathfrak{u}_i(g) \geq \mathfrak{u}_i(g')\quad \forall i \in N \quad \Longrightarrow \quad \mathfrak{w}(\mathfrak{u}(g)) \geq \mathfrak{w}(\mathfrak{u}(g'))$. Moreover, a network g is called *over-connected (under-connected)* with respect to the welfare function $\mathfrak{w}$ if $\exists g' \sqsubset (\sqsupset) g$ such that $\mathfrak{w}(\mathfrak{u}(g')) > \mathfrak{w}(\mathfrak{u}(g))$.

To organize our discussion of the general model, we distinguish between unilateral and bilateral network formation. Network formation is bilateral if a link g_{ij} is determined by both agent i and agent j. Network formation is unilateral if an agent is in control of "his" links. Considering the examples of outcome rules above, examples I,II, and III incorporate bilateral network formation, while outcome rules IV, V, and VI incorporate unilateral network formation. In principle it is also possible to have a model with bilateral formation of directed links.[26] However, we do not consider this unnatural case in the next two subsections.

5.2 Bilateral Network Formation

In this section we focus entirely on the bilateral formation of undirected networks. Thus, a link ij is described by its strength $g_{ij} = g_{ji}$, which is jointly determined by the players i and j, $\psi(s_{ij}, s_{ji}) = g_{ij} = g_{ji}$. Considering the examples of outcome rules above, examples I, II, and III belong to this class of models. In fact, all models we discuss in Section 3 and 4, are examples for bilateral network formation and can be formulated as a Consent Game (outcome rule I). Consistently, the notions of stability we use there, i.e. pairwise stability and pairwise stability with transfers, incorporate bilateral network formation. Now, we want to extend those results to the framework that allows for weighted links, loops and negative links, in addition. For this purpose, let us rewrite the definitions of externalities (Definition 6 and 8):

Definition 14. *A profile of utility functions* $\mathfrak{u}$ *satisfies positive (negative) externalities in* bilateral *network formation if* $\forall g \in G^\Psi, \forall i,j \in N, \forall \delta > 0 : g \mp^{ij} \delta \in G^\Psi$ *and* $\forall k \in N\setminus\{i,j\}$ *it holds that*

$$\mathfrak{u}_k(g \mp^{ij} \delta) \geq (\leq) \mathfrak{u}_k(g).$$

In words: If agents i and j increase the intensity of their link (g_{ij} and g_{ji}), then a player k not involved in this link does not loose (gain) utility. In bilateral network formation players usually have the power to deny links. This is a basic idea in the notion of pairwise stability and it is also true for models based on the outcome rules I and III above. Let us formally define this property:

Definition 15. *An outcome rule* Ψ *satisfies rejection power if* $\forall g, g' \in G^\Psi : g' \sqsubset_i g$ *, it holds that* $\forall s \in \Psi^{-1}(g)$*, there exists* $s'_i \in S_i$ *such that* $\Psi(s_{-i}, s'_i) = g'$.

[26] Note that an outcome rule allowing for a domain of only undirected links cannot be unilateral.

In words: Given a network and a subnetwork that only differ in the intensity of links involving agent i, for any strategy profile inducing the first network, i is able to deviate in order to reach the latter. Simply put, player i is able to deny any intensity that another player proposes toward him.[27] Theorem 1 makes a statement about the bilateral formation of unweighted networks, when positive externalities are satisfied, using the requirement of pairwise stability. With the property of rejection power, we are able to restate this result for general situations of bilateral network formation:

Theorem 6. *Let an outcome rule Ψ satisfy rejection power. If a profile of utility functions $\mathfrak{u}$ satisfies positive externalities in bilateral network formation, then no Nash stable network $g \in G^{\Psi}$ is over-connected with respect to any monotonic welfare function $\mathfrak{w}$.*

Thus reducing the intensity of any set of links in a Nash stable network does not increase the utility of any player. In the special case of unweighted networks (without loops), Theorem 6 has the same implications as Theorem 1 does. In contrast to Theorem 1, it does not need the concavity assumption, but makes a statement about Nash stability instead of pairwise stability. The two concepts do not stand in a simple relation – in general neither $[NS(\mathfrak{u})] \subseteq [PS(\mathfrak{u})]$ nor $[PS(\mathfrak{u})] \subseteq [NS(\mathfrak{u})]$ holds. More precisely, the property of concavity we need for Theorem 1 is sufficient for $[PS(\mathfrak{u})] \subseteq [NS(\mathfrak{u})]$ (as shown in Calvó-Armengol and Ilkiliç, 2007). Both notions share many refinements. One prominent example is the concept of pairwise Nash stability (PNS), also called pairwise equilibrium (Goyal and Joshi, 2006a), which simply incorporates the properties of both Nash equilibrium and pairwise stable networks, $[PNS(\mathfrak{u})] := [NS(\mathfrak{u})] \cap [PS(\mathfrak{u})]$ (see, e.g., Bloch and Jackson, 2006). Since pairwise Nash stability is a refinement of Nash stability, we get a direct corollary: *If the conditions of Theorem 6 are met, then no pairwise Nash stable network is over-connected w.r.t. any monotonic welfare function.*

In Section 3 we presented several examples of unweighted network formation where the results of Theorem 1 and Theorem 6 apply. Besides the examples of unweighted networks, the models of Bloch and Dutta (2005) using outcome rule III satisfy rejection power and positive externalities. The fact that they formulate their model with a budget constraint does not contradict the property of rejection power that we need. Thus, Theorem 6 does apply. However, in this specific example it is not surprising that no Nash stable network can be socially improved by the addition of links because the utility functions are such that *no* network can be improved by the deletion of links, ceteris paribus. The focus of this model – the allocation of effort between different links facing a budget constraint – is not addressed by our result. However, Bloch and Dutta (2005) mention that an interesting variation of their model would be to replace budget constraints by linking costs. Then the individual decisions are not restricted to the allocation of link intensity, but also determine the total amount of link intensity. For such a model our result implies that agents rather under-invest, but surely never over-invest in link intensities due to positive externalities.

A corresponding result to Theorem 6 can be formulated for negative externalities if an outcome rule satisfies some other type of power. Specifically, we need the property that any player can add intensity to any of his links, despite the action of the other player

[27] This property is the special feature of the Consent Game, where both players have to propose a link to have it formed. Similarly, outcome rule III taken from Bloch and Dutta (2005) is based on the minimal intensities announced such that rejection power is satisfied.

involved (the power to form links at free will). One example would be an outcome rule that is based on the maximal effort $g_{ij} = \psi(s_{ij}, s_{ji}) = \max\{s_{ij}, s_{ji}\}$. However, it is not clear in which settings this would be a natural assumption.

5.3 Unilateral Network Formation

Any result so far has concerned undirected links. We finish this paper with a short analysis of the unilateral formation of directed links. Typically, in the formation of directed networks (weighted or unweighted) it is a reasonable assumption that each player is in control of his outgoing links, thus a link $\vec{ij}$ with strength g_{ij} is determined by player i, $g_{ij} = \psi(s_{ij})$.

Among the outcome rules we mentioned, rules IV and V belong to this setting. Outcome rule VI is a special case of unilateral network formation, where agents are in control of their incoming links. Such a model can be easily rewritten to fit into this subsection by changing the role of outgoing and incoming links in the utility function.

Since unilateral network formation does not need any input from the player to whom the link is formed, it is reasonable to reformulate the notion of externalities.

Definition 16. *A profile of utility functions* $\mathfrak{u}$ *satisfies positive (negative) externalities in* unilateral *network formation, if* $\forall g \in G^\Psi, \forall (i,j) \in N \times N, \forall \delta > 0 : g +^{ij} \delta \in G^\Psi$ *and* $\forall k \in N \setminus \{i\}$, *it holds that*

$$\mathfrak{u}_k(g +^{ij} \delta) \geq (\leq) \mathfrak{u}_k(g).$$

In words, positive externalities mean that if a player increases the intensity of one of his outgoing links, then no other player's utility decreases. Here externalities include the player for whom the incoming link has changed, except if the intensity of a loop is changed ($g +^{ii} \delta$). Nash stability is a concept of unilateral deviations. In models of unilateral network formation, we do not need further assumptions to generally characterize the relation between Nash stability and efficiency of directed networks.

Theorem 7. *Let an outcome rule be unilateral* $g_{ij} = \psi(s_{ij})$. *If a profile of utility functions* $\mathfrak{u}$ *satisfies positive (negative) externalities in unilateral network formation, then no Nash stable network is over-connected (under-connected) with respect to any monotonic welfare function* $\mathfrak{w}$.

In words: Given a Nash stable network g. In any subnetwork $g' \sqsubset g$, any player is weakly worse off, $\mathfrak{u}_i(g') \leq \mathfrak{u}_i(g)$.

Consider first the model "A" by Rogers (2006), i.e. outcome rule V. He uses a utility function of the type $\mathfrak{u}_i(g) = \alpha_i + \sum_{j \in N}(\alpha_j + v_j) * g_{ij}$, where each player's utility is recursively defined as the sum of an intrinsic value α_i and the value derived from other agents v_i, which is again a sum of intrinsic value and derived value. In this model there is no tension between stability and efficiency. As Jackson (2008) points out, this relies on several assumptions. Most importantly, Rogers (2006) works with a budget constraint restricting the players choices of how much intensity to "invest" in social relationships. Thus, the efficiency result tells us that the decisions about the allocation of link intensities are efficient. This need not be true for decisions about the amount of intensities if we allow players to decide upon them. Therefore, as in the model of Bloch and Dutta (2005), an interesting variation of Roger's model would be to incorporate linking costs instead of capacity constraints. In such a model, positive externalities are still satisfied since linking costs are internalized. Thus, Theorem 7 implies the inefficient outcomes of such a model are improvable by increasing the intensity of some links, but never by decreasing some intensities.

Now, consider the set of unweighted directed networks, $\{g \in \mathcal{G} : \quad g_{ii} = 0, \forall i \in N; g_{ij} \in \{0,1\}, \forall i,j \in N\}$. Examples for the formation of such networks can be found in Bala and Goyal (2000) (see outcome rule IV). They extensively analyze four models that are called one-way flow model and two-way flow model with or without decay. Let us quickly discuss the interpretation of our result for those models. For concreteness, consider the one-way flow model without decay (OW-model). By $i \rightarrowtail j$ we denote that there exists a directed path from agent i to agent j in a network g, that is a sequence of distinct agents $i_0, i_1, ..., i_T$ such that $i_0 = i$, $i_T = j$ and $\forall t \in \{0, ..., T-1\}$ it holds that $g_{i_t, i_{t+1}} = 1$. Let $\mu_i(g) := \#\{j \neq i : i \rightarrowtail j\}$. Then the one-way flow model is defined as $\mathbf{u}_i^{OW}(g) = \phi(\mu_i(g), d_i(g))$, where ϕ is increasing in the first and decreasing in the second argument. Thus, agents face a trade-off between maintaining costly (outgoing) links and receiving benefits of connections (via outgoing paths).

All four models satisfy positive externalities in unilateral network formation. Clearly, establishing a link $\vec{ij}$ in the OW-model means for some passive player $k \in N \setminus \{i\}$ that his number of outgoing links $d_k(g)$ has not changed, while his number of connections $\mu_k(g)$ might increase. Bala and Goyal (2000) find a large number of Nash stable networks (more than 20,000 for $n = 6$), but only a few of them (two) are strict Nash equilibria. Theorem 7 directly implies that deletion of links in one of the Nash stable networks cannot make any player better off. In any of the four models, the empty network $\vec{g}^{\emptyset} := [0]_{i,j \in N}$ is Nash stable for quite a parameter range, i.e. for $\phi(x+1, x) < \phi(1, 0)$ $\forall x \in \{0, ..., n-1\}$ in the OW-model. However, there might be other Nash stable networks for those parameters, e.g. the wheel network if $\phi(n, 1) > \phi(1, 0)$ is satisfied in addition. Since the empty network is a subnetwork of any other network, Theorem 7 implies that the stable networks have at least as high welfare as the empty network. In the particular case of the OW-model with $\phi(n, 1) > \phi(1, 0)$, the wheel network is uniquely efficient according to utilitarian welfare, as Bala and Goyal (2000) show.

In this model, as in the three other models, there is also some parameter range where the empty network is uniquely efficient, i.e. for $\phi(n, 1) < \phi(1, 0)$. As a trivial implication of Theorem 7, we can conclude that there is no tension (between stability and efficiency) in such a setting because any non-empty network is over-connected and thus must be unstable. In fact, Bala and Goyal (2000) show that myopic best-response dynamics lead to the empty network in various settings. In parameter settings, where the empty network is not efficient, we can interpret these dynamics as leading to an under-connected network due to positive externalities. In the models with two-way flow and/or decay externalities and the consequences of externalities become even more apparent.

6 Concluding Remarks

We have introduced the notion of over-connected and under-connected networks in order to contribute to a better understanding of the tension between stability and efficiency in situations of strategic network formation. A network that is over-connected can be socially improved by the deletion of links; a network that is under-connected can be socially improved by the addition of links. In that way we can relate inefficient outcomes to externalities of link formation.

The basic argument is that positive spillovers/externalities lead to situations where agents are not willing to form links, although it would be collectively beneficial. Negative externalities have the opposite effect: agents form links without internalizing their harm for other agents. Without restricting to any specific network formation model, we have shown the following: for positive externalities, no stable network can be over-connected. For negative externalities no stable network can be under-connected when some other conditions are met. Thus, the inefficient networks in one setting can be improved by the addition of links, while the inefficient networks in the other setting can be improved by the severance of links. Furthermore, we have restated those central results in a more general framework, which includes the formation of weighted networks and the formation of directed networks. Those results are strong enough to exclude that in a stable network any agent can be made better off by decreasing (respectively increasing) the intensity/strength of some links.

Despite their intuitive character, our results are not trivial. It must be noted that externalities are not the only source of inefficiency. Other sources of inefficiencies in the bilateral formation of links are miscoordination of actions in the Consent Game, restrictions on possible deviations, and rejection power in the Consent Game. The first issue can be solved by considering stability concepts that build on pairwise stability as defined in Jackson and Wolinsky (1996). The second, however, is a problem of pairwise stability itself since links are considered one by one. A set of links can have contrary effects than each single link. For example, Theorem 1 needs a concavity assumption to rule out this source of inefficiency that potentially could point into a different direction than externalities. The third case of inefficiency can be illustrated in the example where a player rejects a link although the partner would have benefited heavily from it. This can be ruled out by the introduction of transfers or a property called pairwise sign compatibility (as shown in Theorem 3 and 4). In our work we have not addressed empirical and experimental evidence on the formation of networks with externalities. Kosfeld (2004) provides a survey of network experiments suggesting that, besides individual incentives, considerations of efficiency or "fairness" might also play a role.

In this paper we have shown that positive externalities tend to induce under-connected networks, while negative externalities tend to induce over-connected networks. The contribution of our results is two-fold. Firstly, they shed light into the general tension between stability and efficiency giving a social planner a clear signal in which situations to impede and in which situations it is gainly to promote the formation of relationships. Secondly, the results can be used in specific models to improve the characterization of stable and efficient networks. We have illustrated this with a few examples, while there are many other models of strategic network formation that meet the required conditions. In particular, our results do not rely on assumptions of homogeneous agents or on restrictions to unweighted and undirected networks. We hope that future research will come up with more interesting models accounting for the various nature of social and economic relationships.

APPENDIX

Lemma 2. *(Hellmann (2009))*
The following statements are equivalent:

(1) u is convex (concave) in own links.

(2) u is convex (concave) in own new links.

(3) u is concave (convex) in own current links.

The proof is provided in Hellmann (2009).

Proof of Theorem 1. Let $g \in [PS(u)]$ and suppose that u satisfies positive externalities and concavity. We show that for all $g' \subset g$ it holds that $u_i(g') \leq u_i(g)$ for all $i \in N$. Let $l := l(g, g') = g \setminus g'$ for some $g' \subset g$, and denote $l_i := l_i(g, g') = l \cap L_i(g)$ and $l_{-i} := l \setminus l_i$.

Since g is pairwise stable, all owners of a link prefer to have all their links in g, i.e. $u_i(g) \geq u_i(g \setminus ij)$ for all $j : ij \in l_i$. By Lemma 2 concavity in own links is equivalent to convexity in own current links, and thus it holds: $u_i(g) - u_i(g \setminus l_i) = mu_i(g, l_i) \geq \sum_{ij \in l_i} mu_i(g, ij) \geq 0$. Hence, $u_i(g) \geq u_i(g \setminus l_i)$.

Since u satisfies positive externalities, it holds for $\tilde{g} := g \setminus l_i$ that $u_i(\tilde{g}) \geq u_i(\tilde{g} \setminus l_{-i})$ (because player i does not own a link in l_{-i}), i.e. $l_{-i} \cap L_i(g) = \emptyset$). Therefore: $u_i(g) \geq u_i(g \setminus l_i) \geq u((g \setminus l_i) \setminus l_{-i}) = u(g')$. The same argument holds for all $i \in N$, implying that $w(g) \geq w(g')$ for any welfare function satisfying monotonicity. □

Proof of Theorem 2. Let g be pairwise stable with transfers. We show that for all $g' \subset g$ it holds that $\sum_{i \in N} u_i(g') \leq \sum_{i \in N} u_i(g)$. Suppose that u satisfies positive externalities and concavity. For $g' \subset g$, let $l := l(g, g') := g \setminus g'$ and for each $i \in N$ let $l_i := l_i(g, g') := l \cap L_i(g')$ and $l_{-i} := l_{-i}(g, g') := l \setminus l_i$. Given these definitions we have to show that

$$\sum_{i \in N} u_i(g) - \sum_{i \in N} u_i(g') = \sum_{i \in N} mu_i(g, l) \geq 0. \tag{5}$$

Since u satisfies positive externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \cup l_{-i}). \tag{6}$$

Concavity is equivalent to convexity in own current links which implies for all $i \in N$:

$$mu_i(g, l_i) \geq \sum_{ij \in l_i} mu_i(g, ij). \tag{7}$$

Now, since g is pairwise stable with transfers (2), (6) and (7) imply:

$$\begin{aligned}\sum_{i\in N} mu_i(g,l) &\overset{(6)}{\geq} \sum_{i\in N} mu_i(g,l_i)\\ &\overset{(7)}{\geq} \sum_{i\in N}\sum_{j:ij\in l_i} mu_i(g,ij)\\ &\overset{(*)}{=} \sum_{ij\in l}[mu_i(g,ij)+mu_j(g,ij)] \overset{(2)}{\geq} 0.\end{aligned}$$

To see why equality (*) holds, consider a link $ij \in l$: the link appears only in l_i and l_j and thus $\sum_{k\neq i,j}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right)=0$ and $\sum_{k\in N}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right)=u_i(g\cup ij)-u_i(g)+u_j(g\cup ij)-u_j(g)$. The equality (*) thus holds because $\sum_{ij\in l}\sum_{k\in N}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right)=\sum_{k\in N}\sum_{ij\in l}\left(1_{\{ij\in l_k\}}[u_i(g\cup ij)-u_i(g)]\right)=\sum_{k\in N}\left(\sum_{j:kj\in l_k}[u_i(g\cup ij)-u_i(g)]\right)$. Hence, the sum of marginal utilities of any set of links is non-negative, implying the proposition. □

Proof of Remark 1. Let $g\in\tilde{G}$ such that it is stable according to one of the stability concepts, i.e. $g\in[PS(u)]\cup[PS^t(u)]$. Since u satisfies the assumptions of Theorem 1 and Theorem 2 on $\tilde{G}$, we can show equivalently to both proofs that for all $\bar{g}\subset g$ it holds that $w^u(\bar{g})\leq w^u(g)$, since any $\bar{g}\subset g$ is by assumption contained in $\tilde{G}$. □

Proof of Lemma 1. By Lemma 2 it suffices to show that u^{CO} satisfies convexity in own current links, that is $\forall i\in N,\forall g\in G,$ and $\forall l\subseteq L_i(g)$, it holds that $mu_i^{CO}(g,l)\geq\sum_{ij\in l} mu_i^{CO}(g,ij)$.

Denote $\kappa_i(g,l):=\{k\in N:\ d_{ik}(g)<d_{ik}(g\setminus l)\}$ as the set of agents whose distance to agent i increases when deleting the set of links l from network g. Since distances cannot decrease when deleting links, we can rewrite marginal utility in the following way:

$$\begin{aligned}mu_i^{CO}(g,l) &= w_{ii}+\sum_{k\neq i}\delta^{d_{ik}(g)}w_{ik}-\sum_{k:ik\in g}c_{ik}-[w_{ii}+\sum_{k\neq i}\delta^{d_{ik}(g\setminus l)}w_{ik}-\sum_{m:im\in g\setminus l}c_{im}]\\ &= \sum_{k\in\kappa_i(g,l)}(\delta^{d_{ik}(g)}-\delta^{d_{ik}(g\setminus l)})w_{ik}-\sum_{ij\in l}c_{ij}.\end{aligned}$$

Now, consider some network g, some player i and some set of player i's links $l\subseteq L_i(g)$. Suppose that $|l|\geq 2$.[28]

To show the claim, let us assume the contrary, i.e. $mu_i^{CO}(g,l)<\sum_{ij\in l} mu_i^{CO}(g,ij)$.

[28]For $|l|<2$ the claim $mu_i(g,l)\geq\sum_{ij\in l} mu_i(g,ij)$ trivially holds.

$$\begin{aligned} mu_i(g,l) &< \sum_{ij\in l} mu_i(g,ij) \\ \sum_{k\in\kappa_i(g,l)} (\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash l)})w_{ik} - \sum_{ij\in l} c_{ij} &< \sum_{ij\in l} [\sum_{k\in\kappa_i(g,ij)} (\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash ij)})w_{ik} - c_{ij}] \\ \sum_{k\in\kappa_i(g,l)} (\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash l)})w_{ik} &< \sum_{ij\in l}\sum_{k\in\kappa_i(g,ij)} (\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash ij)})w_{ik} \qquad (8) \end{aligned}$$

To see that Eq. 8 cannot hold, note the following three properties of geodesic distances that were already used in Calvó-Armengol and Ilkiliç (2007):

1. $\forall k \in N$ and $\forall ij \in l$, it holds that $(\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash l)})w_{ik} \geq (\delta^{d_{ik}(g)} - \delta^{d_{ik}(g\backslash ij)})w_{ik}$.
2. For all $ij, im \in l$, it holds that $\kappa_i(g,ij) \cap \kappa_i(g,im) \ = \emptyset$.
3. $\bigcup_{ij\in l} \kappa_i(g,ij) \ \subseteq \kappa_i(g,l)$.

Thus, we conclude that $mu_i(g,l) \geq \sum_{ij\in l} mu_i(g,ij)$. □

Proposition 4. *In the symmetric connections model, g^{Tetra} is under-connected with respect to the utilitarian welfare function for any parameters δ and c, for which g^{Tetra} is pairwise stable.*

Proof. We have to show that if δ and c are such that $g^{Tetra} \in PS(u_{\delta,c})$, then $\exists g' \supset g^{Tetra}$ for which $w_{\delta,c}(g') > w_{\delta,c}(g^{Tetra})$. Specifically, we show that the condition

$$c \leq \delta - \delta^8 + \delta^2 - \delta^7 + \delta^3 - \delta^6 + 2(\delta^4 - \delta^5) := ub \qquad (9)$$

is necessary for stability, but sufficient for $w_{\delta,c}(g^{Tetra} \cup 26) > w_{\delta,c}(g^{Tetra})$. The label of the players correspond to figure 1.

The first part was done in Jackson and Wolinsky (1996) already. Suppose that $c > ub$, then player 1 benefits from cutting 12 (because his change in benefits is just ub).

For the second part denote by $\beta_i := \sum_{j\neq i} \delta^{d_{ij}(g^{Tetra}\cup 26)} - \sum_{j\neq i} \delta^{d_{ij}(g^{Tetra})}$ the marginal benefits for player i and by $\Delta := \sum_{i\in N} \beta_i$ the sum of marginal benefits. This allows to write

$$w_{\delta,c}(g^{Tetra} \cup 26) > w_{\delta,c}(g^{Tetra}) \Longleftrightarrow \Delta > 2c. \qquad (10)$$

It is straightforward to derive that

$$\begin{aligned} \beta_1 &= \beta_8 = \delta^2 - \delta^4 + \delta^3 - \delta^4 \\ \beta_2 &= \beta_6 = \delta - \delta^5 + \delta^2 - \delta^4 + \delta^2 - \delta^5 + 2(\delta^3 - \delta^4) \\ \beta_3 &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5 \\ \beta_4 &= \beta_9 = \beta_{10} = \beta_{11} = \delta^3 - \delta^4 \\ \beta_7 &= \delta^2 - \delta^5 + \delta^3 - \delta^4 + \delta^3 - \delta^5, \end{aligned}$$

and $\beta_i = 0$ for all other i.

This yields

$$\Delta = 2(\delta - \delta^5) + 4(\delta^2 - \delta^4) + 4(\delta^2 - \delta^5) + 12(\delta^3 - \delta^4) + 2(\delta^3 - \delta^5). \tag{11}$$

To show that $\Delta > 2c$ under the condition $c \leq ub$, it is sufficient to show that $\Delta > 2ub$ holds. Recall that,

$$2ub(g) = 2(\delta - \delta^8) + 2(\delta^2 - \delta^7) + 2(\delta^3 - \delta^6) + 4(\delta^4 - \delta^5). \tag{12}$$

Thus,

$$\Delta > 2ub \iff 6\delta^2 + 12\delta^3 - 20\delta^4 - 4\delta^5 + 2\delta^6 + 2\delta^7 + 2\delta^8 > 0 \tag{13}$$

Numerically it can be checked that (13) holds for all $\delta \in (0, 1)$ (we used Maple). □

Proof of Theorem 3. Let g be pairwise stable with transfers. We show that for all $g' \supset g$ it holds that $\sum_{i \in N} u_i(g') \leq \sum_{i \in N} u_i(g)$. Suppose that u satisfies negative externalities and concavity. For $g' \supset g$, let $l := g' \setminus g$ and for each $i \in N$ let $l_i = l \cap L_i(g')$ and $l_{-i} := l \setminus l_i(g, g')$. Since u satisfies negative externalities, it holds for all $i \in N$ that:

$$u_i(g') \leq u_i(g' \setminus l_{-i}). \tag{14}$$

Concavity is equivalent to concavity in own new links, which implies for all $i \in N$:

$$u_i(g \cup l_i) - u_i(g) \leq \sum_{j:ij \in l_i} u_i(g \cup ij) - u_i(g). \tag{15}$$

Now, since g is pairwise stable with transfers, (14) and (15) imply:

$$\begin{aligned}
\sum_{i \in N}(u_i(g') - u_i(g)) &= \sum_{i \in N}(u_i(g \cup l_i \cup l_{-i}) - u_i(g)) \\
&\stackrel{(14)}{\leq} \sum_{i \in N}(u_i(g \cup l_i) - u_i(g)) \\
&\stackrel{(15)}{\leq} \sum_{i \in N}\Big(\sum_{j:ij \in l_i}[u_i(g \cup ij) - u_i(g)]\Big) \\
&\stackrel{(*)}{=} \sum_{ij \in l} u_i(g \cup ij) - u_i(g) + u_j(g \cup ij) - u_j(g) \stackrel{(2)}{\leq} 0,
\end{aligned}$$

where the equality (*) holds because for each link $ij \in l$ it holds that $ij \in l_1(k)$ if and only if $k \in \{i, j\}$ and only links in l are considered. □

Proof of Theorem 4. We show first $[PS(u)] \subseteq [PS^t(u)]$, and then $[PS(u)] \supseteq [PS^t(u)]$.

'⊆' Let g be pairwise stable. By pairwise stability we get for all $ij \in g$ that $u_i(g \setminus ij) \leq u_i(g)$ and $u_j(g \setminus ij) \leq u_j(g)$, hence $u_i(g \setminus ij) + u_j(g \setminus ij) \leq u_i(g) + u_j(g)$. Further it holds by pairwise stability for all $ij \notin g$ that either $u_i(g) = u_i(g \cup ij)$ and $u_j(g) = u_j(g \cup ij)$ or for one of the two involved players i or j: $u_i(g) > u_i(g \cup ij)$. In the latter case PSC implies also $u_j(g) > u_j(g \cup ij)$, hence $u_i(g \cup ij) + u_j(g \cup ij) \leq u_i(g) + u_j(g)$, implying that g is pairwise stable with transfers.

'⊇' Let g be pairwise stable with transfers. Then it holds for all $ij \in g$ that $u_i(g \setminus ij) + u_j(g \setminus ij) \leq u_i(g) + u_j(g)$, hence $u_i(g \setminus ij) \leq u_i(g)$ for at least one of the two involved players. By PSC it has to hold that also $u_j(g \setminus ij) \leq u_j(g)$, satisfying the first condition for pairwise stability. Further, pairwise stability with transfer implies for all $ij \notin g$ that $u_i(g \cup ij) + u_j(g \cup ij) \leq u_i(g) + u_j(g)$. Thus $u_i(g \cup ij) \leq u_i(g)$ for at least one of the two players. By PSC however this has to hold also for the other player, i.e. $u_j(g \cup ij) \leq u_j(g)$. Hence, g is also pairwise stable. □

Proof of Remark 2. Let $g \in \tilde{G}$ be pairwise stable with transfers. Since for all $g' \supset g$ it holds that $g' \in \tilde{G}$, and u satisfies the assumptions of Theorem 3 on $\tilde{G}$, we can show equivalently to the proof of Theorem 3 that for all $\bar{g} \supset g$ it holds that $w^u(\bar{g}) \leq w^u(g)$. We get the analogous result for pairwise stability if PSC is satisfied by Theorem 4. □

Proof of Theorem 5. We have to show that for all $g' \supset g$ the following holds:

$$\exists i \in N: \quad u_i(g') > 0 \Rightarrow \exists j \in N: \quad u_j(g') < u_j(g).$$

Similarly to the above proofs, for $g' \supset g$ let $l = g' \setminus g$, $l_i = l \cap L_i(g')$ and $l_{-i} = l \setminus l_i$. Let $N(l) := \{i \in N : l_i \neq \emptyset\}$ be the set of players who are involved in at least one link in l. Suppose that $u_k(g') = u_k(g)$ for all $k \notin N(l)$ (otherwise the result is immediately established because of non-positive externalities). Similarly to the proof of Theorem 3, concavity and negative externalities imply for all $i \in N$:

$$u_i(g') - u_i(g) \overset{(14)}{\leq} u_i(g \cup l_i) - u_i(g) \overset{(15)}{\leq} \sum_{j:ij \in l_i} u_i(g \cup ij) - u_i(g). \tag{16}$$

Thus, for g' to be Pareto preferred to g there has to exist an $i : u_i(g') > u_i(g)$ and thus because of (16) there has to exist $i_1 \in N$: $u_i(g \cup ii_1) > u_i(g)$. But then because of pairwise stability of g: $u_{i_1}(g \cup ii_1) < u_{i_1}(g)$. However, for i_1 to have $u_{i_1}(g') \geq u_{i_1}(g)$, there has to exist an i_2 such that $u_i(g \cup i_1 i_2) > u_i(g)$, since (16) holds and already $u_{i_1}(g \cup ii_1) < u_{i_1}(g)$. Continuing in this manner, for g' to be Pareto preferred to g, there has to exists sequence $(i_k)_{k=1,...,K}$ of pairwise distinct players $i_k \in N(l)$ such that $u_{i_k}(g \cup i_k i_{k+1}) > u_{i_k}(g)$. By pairwise stability of g, $mu_{i_k}(g \cup i_k i_{k+1}, i_k i_{k+1}) > 0$ implies $mu_{i_{k+1}}(g \cup i_{k+1} i_k, i_{k+1} i_k) < 0$ for all $k = 1, ..., K-1$. Since transitivity of negative marginal utility in new links holds, and $mu_{i_{k+1}}(g \cup i_{k+1} i_k) < 0$ for all $k = 1, ..., K-1$, we get that $mu_{i_k}(g \cup i_k i_j) < 0$ for all $j < k$, and thus $i_k \notin \{i, i_1, ..., i_{k-1}\}$, and hence $K \leq |N(l)|$. But since $N(l)$ is finite, we get for the last player i_K in the sequence:

$$u_{i_K}(g') - u_{i_K}(g) \overset{(14)}{\leq} u_{i_K}(g \cup l_{i_K}) - u_{i_K}(g) \overset{(15)}{\leq} \sum_{j:i_K j \in l_{i_K}} u_{i_K}(g \cup ij) - u_{i_K}(g) < 0,$$

since $u_{i_K}(g \cup ij) - u_{i_K}(g) < 0$ for all $j \in \{i, i_1, ..., i_{K-1}\}$. Thus, if there exists a $i \in N$ such that $u_i(g') > u_i(g)$, then there has to exist a player j such that $u_j(g') < u_j(g)$, completing the proof. □

Proof of Corollary 3. We show here the obvious, that a regular pairwise stable network is also pairwise stable with transfers, the remaining is implied by Theorem 3. If the utility function satisfies (4) and the network is regular, then all players receive the same utility since all arguments of the utility functions are equal. Alike are the marginal utilities of any two involved players from forming a link. Since g is pairwise stable, we have for any $i, j \in N$: $mu_i(g \cup ij, ij) = mu_j(g \cup ij, ij) \leq 0$, and thus $mu_i(g \cup ij, ij) + mu_j(g \cup ij, ij) \leq 0$, implying pairwise stability with transfers. In the star for any two peripheral players same considerations hold. □

Proof of Proposition 1. All pairwise stable networks consist of completely connected components that can be ordered according to size such that each larger component of size m satisfies $m > n^2$, where n is the size of the smaller component. There cannot be singleton components, since each player is better off connecting to some player than to none, and each player i wants a link to a player j, for whom $d_j \leq d_i$. Note that this implies that there exists at least one component of size 3 if $n \geq 3$. Since any even sized network of $n/2$ separate pairs and any odd sized network of $(n-2)/2$ pairs and the remaining three players being connected by 2 links is strongly efficient, it is also component efficient for any component of size n and, hence, strictly welfare better than any completely connected component of at least size 3. For the exact calculations see Jackson and Wolinsky (1996). Hence any completely connected component of size 3 or larger contains a welfare better subcomponent, whereas a completely connected component of size 2 is component welfare maximizing, implying the result. □

Proof of Proposition 2. Let $c < \frac{\rho}{(\rho+2k+2)(\rho+2k)}$, then for the welfare maximizing number of links it holds that $1/2D^*(g) \geq k$. Since any network, which contains $1/2D^*(g)$ links is welfare maximizing, any network, which has less than $1/2D^*(g)$ links is under-connected. By Theorem 3 no pairwise stable network can be under-connected since u^{PR} satisfies negative externalities and concavity. Thus, any network $g \in [PS^t]$ has to contain at least $1/2D^*(g) \geq k$ links. □

Proof of Theorem 6. Given some outcome function Ψ that satisfies rejection power, let there be a society $(N, G^\Psi, \mathfrak{u})$ such that $\mathfrak{u}$ satisfies positive externalities in bilateral network formation. Let $g \in [NS(\mathfrak{u})]$ and let $g' \sqsubset g$. We show that then $\mathfrak{u}_i(g') \leq \mathfrak{u}_i(g)$ for all $i \in N$.

For each $i \in N$ let $g^i \in G^\Psi$ be such that $g^i_{kl} = g_{kl}$ for all $k, l \neq i$ and $g^i_{kl} = g'_{kl}$ otherwise. Note that $g^i \sqsubset_i g$. Since Ψ satisfies rejection power, we get that each i can change strategies such that g^i is induced. By Nash stability we get that $\mathfrak{u}_i(g) \geq \mathfrak{u}_i(g^i)$.

It remains to show that for all i, $\mathfrak{u}_i(g^i) \geq \mathfrak{u}_i(g')$. For all $k, l \in N$ let $\delta_{kl} = g^i_{kl} - g'_{kl}$ be the differences in link intensity between g^i and g'. Note that by construction of g^i it holds

$\delta_{kl} = 0$ if $k = i$ or $l = i$ and $\delta_{kl} \geq 0$, since $g' \sqsubset g$. By positive externalities, we get that $\mathfrak{u}_i(g^i) \geq \mathfrak{u}_i(g')$, since $g^i = g'(\mp^{kl})_{k,l \in N}\delta_{kl}$ and $\delta_{kl} = 0$ if $k = i$.

Thus, $\mathfrak{u}_i(g) \geq \mathfrak{u}_i(g^i) \geq \mathfrak{u}_i(g')$ for all $i \in N$ implying that $\mathfrak{w}(g) \geq \mathfrak{w}(g')$ for any welfare function satisfying monotonicity. □

Proof of Theorem 7. Let the outcome rule Ψ be unilateral such that $g_{ij} = \psi(s_{ij})$ for all $i, j \in N$. We show the result for positive externalities, while the proof for negative externalities works identically. Let $g \in [NS(\mathfrak{u})]$. We show that for all $g' \in G^\Psi$ such that $g' \sqsubset g$, it holds that $\mathfrak{u}_i(g') \leq \mathfrak{u}_i(g)$ for all $i \in N$. Let $\tilde{g} \in G^\Psi$ such that $\tilde{g}_{kl} = g_{kl}$ for all $k \neq i$ and $\tilde{g}_{kl} = g'_{kl}$ if $k = i$.

Step 1: Since g is Nash stable, there exists $s \in \Psi^{-1}(g)$ such that s is a Nash equilbrium in the corresponding game Γ. Let $\tilde{s}_i := (\tilde{s}_{ij})_{j \in N}$ with $\tilde{s}_{ij} \in \psi^{-1}(g'_{ij})$ (that is a strategy of player i that creates all his outgoing links as in g'). By construction it holds that $(s_{-i}, \tilde{s}_i) \in \Psi^{-1}(\tilde{g})$. Since s is a Nash equilibrium, it holds that $\mathfrak{u}_i(\Psi(s)) \geq \mathfrak{u}_i(\Psi(s_{-i}, \tilde{s}_i)) \Leftrightarrow \mathfrak{u}_i(g) \geq \mathfrak{u}_i(\tilde{g})$.

Step 2: It remains to show that $\mathfrak{u}_i(\tilde{g}) \geq \mathfrak{u}_i(g')$. This is however directly implied by positive externalities, since $\tilde{g} = g'(+^{kl})_{k,l \in N}\delta_{kl}$ such that $\delta_{kl} = 0$ if $k = i$ and $\delta_{kl} \geq 0$ for all $k, l \in N$ (by the construction of $\tilde{g}$).

Thus, we have shown that $\mathfrak{u}_i(g) \overset{step1}{\geq} \mathfrak{u}_i(\tilde{g}) \overset{step2}{\geq} \mathfrak{u}_i(g')$. The same argument holds for all $i \in N$, implying that $\mathfrak{w}(g) \geq \mathfrak{w}(g')$ for any welfare function satisfying monotonicity. □

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