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Keywords: sender-receiver games, strategic information transmission
JEL codes: D83

1 Introduction

Crawford and Sobel's seminal paper (1982) concerning one-shot sender-receiver games is an essential reference for most of the literature in strategic information transmission. In particular, multi-stage games often rely on the uniqueness of per-stage equilibrium solutions. However, Crawford and Sobel substantially assume that equilibria are unique.

In particular, Crawford and Sobel (1982) consider the following one-shot game of strategic communication. The payoff of two agents, N and E, depend on action a and the true state of the world ω. Agent N has prior beliefs about the state of the world, that are represented by a non-degenerate distribution function. Instead, agent E can observe the true state perfectly. First, agent E sends a message to agent N, then agent N chooses action a and the payoff are realized. Crawford and Sobel show that Nash Bayesian equilibria are partitional: agent E will introduce noise into his signal so that only one action will be implemented for all the states that belong to the same element of the equilibrium

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partition. Moreover, equilibrium partitions will have finite cardinality, i.e. the state space will be partitioned into a finite number of proper subsets.

Crawford and Sobel impose a particular monotonicity condition on the equilibrium solutions. That condition implies that a unique equilibrium partition with cardinality I can exist. Moreover, they consider only uniform message rules, i.e. agent E will choose a message from the equilibrium subset M_i randomly, according to a uniform distribution, if the true state lies in the equilibrium subset Ω_i.

The paper shows that Perfect Bayesian equilibria need not be unique for the game of Crawford and Sobel under two different respects. First, an equilibrium partition with cardinality I need not be unique. In particular, provided an equilibrium partition of the state space with I elements is unique under some prior distribution function, that partition can be shown to belong to the non-singular subset of equilibrium partitions with the same size under other distribution functions. Consequently, there can be different equilibrium action profiles for partitions with the same cardinality.

Second, Perfect Bayesian equilibria need not be unique because the equilibrium message rule and, hence, the equilibrium beliefs are not unique. In particular, provided a Perfect Bayesian equilibrium exists under some prior distribution function with a unique equilibrium profile of actions and a particular equilibrium message rule, there will be other Perfect Bayesian equilibria under the same distribution function with the same equilibrium profile of actions but different message rules and equilibrium beliefs.

2 Set-up

Assumption 1 *: the payoffs of the agents N and E depend on the action a and the state of nature ω. Action a belongs to the action space A, that is an interval of real numbers, while the state space Ω is the closed unit interval on the real line. In particular, agent N has a twice continuously differentiable von Neumann-Morgenstern utility function $U^N(a,\omega)$; agent E has a twice continuously differentiable von Neumann-Morgenstern utility function $U^E(a,\omega,b)$, where b is a scalar parameter. The utility functions are such that:*

$$\text{given } \omega,\ \exists a \in A : U_a^i(\bullet) = 0,\ \text{with } i = N, E \tag{1}$$

$$U_{aa}^{\,i}(\bullet) < 0 \qquad \vee\, a \in A,\ \text{with } i = N, E \tag{2}$$

$$U_{a\omega}^i(\bullet) > 0 \qquad \text{with}\ \ i = N, E \tag{3}$$

From $(1)-(2)$, given ω, $U^i(\bullet)$ has a unique maximum in a for $i = N, E$. Let:

$$a_\omega^i = \arg\max_a U^i(\bullet)\,,\ i = N, E \tag{4}$$

Parameter b in agent E's utility function is a measure of the conflict of interest between the agents. In particular:

Assumption 2 *: the scalar parameter b is such that in* (4) *the best action* a_ω^E *for a perfectly informed agent E is always lower than the best action* a_ω^N *for a perfectly informed agent* N[1]*. Only agent E observes the true state of nature. Instead, agent N has prior beliefs about the state of nature that are represented by the distribution function (d.f.)* $F(\omega)$*, with continuous probability density function* $f(\omega)$ *such that* $f(\omega) > 0$ *for every* ω *in* $(0,1)$[2]*.*

First, agent E observes ω*, then he chooses and sends one message m to agent N. The cardinality of the set M of messages is not lower than the cardinality of* Ω*. Agent N receives message m and, then, he chooses one action a in A. Once the action is selected by agent N, the payoffs are realized.*

All aspects of the game except ω *are common knowledge.*

Agent E chooses a message rule, i.e. a set of generalized probability density functions, denoted $\{\varphi(m \mid \omega)\}$, with cardinality $|\Omega|$, such that $\int_M \varphi(m \mid \omega)dm = 1$ for each state ω. Agent N chooses an action rule, i.e. a set of generalized probability density functions, denoted $\{\alpha(a \mid m)\}$, with cardinality $|M|$, such that $\int_A \alpha(a \mid m)da = 1$ for each message m. Let $\rho(\omega \mid m)$ denote agent N's probability density function of ω conditional on having received message m. A Perfect Bayesian equilibrium is a pair of message rule $\{\varphi^*(m \mid \omega\})$ and action rule $\{\alpha^*(a \mid m)\}$, and a system of beliefs $\{\rho^*(\omega \mid m)\}$ such that:

1) the equilibrium message rule maximizes agent E's expected payoff for every state ω given the equilibrium action rule;

2) the equilibrium action rule maximizes agent N's expected payoff for every message m where the expectation satisfies the following condition:

$$\begin{aligned} \Omega_m^* &= \{\omega \mid \varphi^*(m \mid \omega) > 0\} \neq \emptyset \rightarrow \qquad (5) \\ \rho^*(\omega \mid m) &= \frac{\varphi^*(m \mid \omega) f(\omega)}{\int_{\Omega_m^*} \varphi^*(m \mid \omega) f(\omega) d\omega} \ \ \forall\, \omega \in \Omega_m^*;\ \rho^*(\omega \mid m) = 0 \ \ \forall\, \omega \notin \Omega_m^* \end{aligned}$$

3 Results

Proposition 1 is Theorem 1 in Crawford and Sobel (1982, p.1437) adapted to Perfect Bayesian Equilibria, instead of Bayesian Nash equilibria.

Proposition 1 *:*

1) every Perfect Bayesian Equilibrium is partitional, i.e.:

- there exists a partition of M into I^* *disjoint subsets* M_i^*, $i = 1, ...I^*$.

- There exists a partition of Ω *into* I^* *subsets denoted* Ω_i^* *such that* $\Omega_i^* = [\underline{\omega}_i^*, \overline{\omega}_i^*]$ *with* $\underline{\omega}_i^* < \overline{\omega}_i^*$ *for* $(I^* - 1)$ *elements of the partition at least; the equilibrium message rule is such that:*

$$\int_{M_i^*} \varphi^*(m \mid \omega \in \Omega_i^*)dm = 1; \int_{M_i^*} \varphi^*(m \mid \omega \notin \Omega_i^*)dm = 0$$

[1] All the results hold for the analogous case in which a_ω^E is always greater than a_ω^N.

[2] If the density function $f(\omega)$ were nil for non-empty subsets of the state space, then there could be different partitions of the state space substantially equivalent with respect to the induced action profile, given the prior beliefs.

- There exists a profile of I^* actions denoted a_i^*, is such that:

$$a_i^* = \arg\max_{a \in A} \int_{\Omega} U^N(a,\omega)\rho^*(\omega \mid m \in M_i^*)\, d\omega$$

given (5).

2) Every equilibrium partition has a finite number $I^*$of elements.

3) If there exists an equilibrium partition with cardinality $I^* > 1$, then there will be an equilibrium partition with cardinality $(I^* - 1)$.

Given a d.f. $H(\omega)$, let a_H denote the unique action such that:

$$a_H = \arg\max_{a} \int_{\Omega} U^N(\bullet)\, dH(\omega) d\omega \tag{6}$$

Moreover, let P^*_{H,I^*} denote the equilibrium partition of the state space with cardinality I^* under the d.f. $H(\omega)$.

Crawford and Sobel (1982, p.1444) assume that the following monotonicity condition holds in equilibrium:

Condition 1 *M: given $P^*_{F,I^*} = \{\Omega_i^*\}$ and $P^*_{F,\tilde{I}^*} = \left\{\tilde{\Omega}_i^*\right\}$, if $\overline{\omega}^*_{1,I^*} > \overline{\omega}^*_{1,\tilde{I}^*}$, then $\overline{\omega}^*_{i,I^*} > \overline{\omega}^*_{i,\tilde{I}^*}$ for all $i \geq 2$.*

Condition M implies that there will be a unique equilibrium partition for each cardinality of the partitions: if P^*_{F,I^*} and $P^*_{F,\tilde{I}^*}$ exist, then I^* will be lower than $\tilde{I}^*$.

The paper shows that the one-to-one correspondence between cardinality and equilibrium partition will not hold if Crawford and Sobel's monotonicity condition is dropped. The following Corollary and Lemma are useful in order to prove that multiple equilibrium partitions with the same cardinality can exist. In particular, Lemma 1 shows that identical action profiles will be induced under prior distribution functions defined on the same support and having equal mean, provided $U^N_{a\omega\omega}(\bullet)$ is equal to zero[3].

Corollary 1 *: $\left[U^E(a_F, 1, b) - U^E\left(a_0^N, 1, b\right)\right] > 0$ is a necessary condition for the existence of P^*_{F,I^*} with $I^* > 1$.*

Lemma 1 *: provided $U^N_{a\omega\omega}(\bullet) = 0$, then a_H will be equal to a_K if the d.f. $H(\omega)$ and the d.f. $K(\omega)$ have the same mean.*

Proposition 2 proves in the following way that Perfect Bayesian equilibria need not be unique. Suppose that under the prior d.f. in the following way. there exists a unique P^*_{F,I^*} with the correspondingly unique equilibrium action profile $\overline{a}_{F,I^*}$. Consider another d.f. $G(\omega)$ that both satisfies the marginal likelihood ratio property with respect to the prior d.f. $F(\omega)$, and induces a unique P^*_{G,I^*}

[3]The condition $U^N_{a\omega\omega}(\bullet) = 0$ is satisfied by a class of commonly used utility functions. Crawford and Sobel (1982, p. 1440)) assume that $U^N(a,\omega) = -(a-\omega)^2$ in an example.

with unique equilibrium action profile $\overline{a}_{G,I^*}$. First-order stochastic dominance between the distribution functions and MLRP will imply a complete ordering of the upper and lower bounds of all the subsets in P^*_{F,I^*} and P^*_{G,I^*}. Given that ranking, there will exist another d.f. $Y(\omega)$, that is a mixture of distribution functions, under which the set of equilibrium partitions with cardinality I^* contains both P^*_{F,I^*} and P^*_{G,I^*}.

Proposition 2 *: suppose that there exists a unique P^*_{F,I^*} with $I^* > 1$. Provided $U^N_{a\omega\omega}(\bullet) = 0$, there exist other distribution functions $Y(\omega)$ such that the set of P^*_{Y,I^*} is non-singular and contains P^*_{F,I^*}.*

Given Proposition 2 there can be different equilibrium action profiles notwithstanding the same cardinality of the equilibrium partitions: under the d.f. $Y(\omega)$ both $\overline{a}_{F,I^*}$ and $\overline{a}_{G,I^*}$ will be equilibrium action profiles.

Proposition 3 shows that an equilibrium message rule is not unique even in correspondence to a unique P^*_{F,I^*}. In particular, represent agent N's posterior beliefs as distribution functions. If an equilibrium message rule supports a posterior d.f. $H(\omega)$, then all the message rules inducing posterior distribution functions with the same mean of $H(\omega)$ and rankable with respect to $H(\omega)$ according to second-order stochastic dominance will be equilibrium message rules.

Proposition 3 *: suppose that there exists a unique P^*_{F,I^*}. Provided $U^N_{a\omega\omega}(\bullet) = 0$, the equilibrium message rule and beliefs are not unique under the d.f. $F(\omega)$.*

4 Conclusions

The paper shows that Perfect Bayesian equilibria need not be unique in the strategic communication game of Crawford and Sobel (1982). Consequently, there is not a correspondence which associates with each cardinality of the equilibrium partitions one and only one equilibrium action profile for every prior distribution function. Moreover, the equilibrium message rules and beliefs are not unique for given prior beliefs.

Non-uniqueness of one-shot equilibria can be relevant for multi-stage games of strategic communication.

5 Appendix

Proof. of Proposition 1

Step 1. From strict concavity in (2), there will be a unique action that maximizes agent i's expected payoff function for each d.f. $F(\omega)$. Hence, agent N will never use mixed strategies, whatever his beliefs $\{\rho(\omega \mid m)\}$. Agent N's action rule will be a function $\tilde{a}(m) : M \to A_M$ where:

$$A_M = \left\{ a_m \mid \int_\Omega U^N_a(a_m, \omega)\, \rho(\omega \mid m) d\omega = 0 \right\} \tag{7}$$

Suppose that the d.f. $H(\omega)$ dominates the d.f. $K(\omega)$ in the sense of first order stochastic dominance. Let a_H and a_K denote the action levels such that:

$$\int_\Omega U_a^i(a_H, \bullet)dH(\omega)d\omega = \int_\Omega U_a^i(a_K, \bullet)dK(\omega)d\omega = 0 \tag{8}$$

From (3), (8) and (2) it follows that:

$$\int_\Omega U_a^i(a_H, \bullet)dH(\omega)d\omega > \int_\Omega U_a^i(a_H, \bullet)dK(\omega)d\omega \to a_H > a_K \tag{9}$$

From (9) the best value of a for the fully informed agent i in (4) is a continuous, strictly monotonic function of the true value of ω, i.e.:

$$a_\omega^i > a_{\omega'}^i \longleftrightarrow \omega > \omega' \qquad \text{with } i = N, E \tag{10}$$

Let A^i be the set of the a_ω^i in (4). Given (1):

$$A^i = \left[a_0^i, a_1^i\right] \subseteq A \quad \text{with } i = N, E; \; \left|A^i\right| = |\Omega| \qquad \text{with } i = N, E \tag{11}$$

From (7) and (11):

$$A_M \subseteq A^N \tag{12}$$

Let:

$$I = |A_M| \tag{13}$$

Consequently, $I \leq \min\left\{|M|, \left|A^N\right|\right\}$.

Step 2. Rank all the elements in A_M in the following way:

$$A_M = \{a_1, ..., a_i, ..., a_I \mid a_{i-1} < a_i < a_{i+1}\}$$

Let:

$$M_i = \{m \mid \tilde{a}(m) = a_i\} \quad \text{with } i = 1, ..., I$$

$$\begin{aligned} \Omega_i &= \left\{\omega \mid U^E(a_i, \omega, b) \geq U^E(a_j, \omega, b) \quad \vee\, a_j \in (A_M \setminus a_i)\right\} \\ \text{with } i &= 1, ..., I \end{aligned}$$

By construction, $\bigcup_{i=1}^I \Omega_i = \Omega$. From (2):

$$\begin{aligned} \omega &\in \Omega_i \cap \Omega_j \to \omega \notin \bigcup_{p \neq i,j} \Omega_p \\ a_i &> a_j \text{ and } \omega \in \Omega_i \cap \Omega_j \to a_j < a_\omega^E < a_i \end{aligned} \tag{14}$$

From (14), if $\omega \in \Omega_i \cap \Omega_j$ and $a_j < a_{i-1}$, then either $a_\omega^E \in (a_j, a_{i-1}]$ and $\omega \notin \Omega_i$, or $a_\omega^E \in [a_{i-1}, a_i)$ and $\omega \notin \Omega_j$, that is contradictory. Hence:

$$\omega \in \Omega_i \to \omega \notin \Omega_j \quad \vee\, j < i - 1, \vee j > i + 1 \tag{15}$$

From (3) the utility functions have increasing differences in (a, ω) (Milgrom-Shannon (1994)), i.e.:

$$U^i(a, \omega, \bullet) - U^i(a', \omega, \bullet) > U^i(a, \omega', \bullet) - U^i(a', \omega', \bullet) \quad \vee\, a > a\prime,\, \omega > \omega\prime \tag{16}$$

From (16):

$$\begin{aligned} \Omega_i \cap \Omega_j &\neq \emptyset \to |\Omega_i \cap \Omega_j| = 1 \\ a_i &> a_j \text{ and } \omega \in \Omega_i \text{ and } \omega' \in \Omega_j \to \omega > \omega' \quad \vee \omega \neq \omega' \end{aligned} \tag{17}$$

Finally from (15) and (17):

$$a_i > a_p > a_j \text{ and } \Omega_i, \Omega_j \neq \emptyset \to \Omega_p \neq \emptyset \tag{18}$$

From (14) – (18):

$$\begin{aligned} \Omega_i &\neq \emptyset \to \Omega_i = [\underline{\omega}_i, \overline{\omega}_i] \quad \text{with } \underline{\omega}_i \leq \overline{\omega}_i \\ \omega &\in \Omega_i \to a_{i-1} < a^E_\omega < a_{i+1} \\ \Omega_{i-1}, \Omega_i &\neq \emptyset \to \overline{\omega}_{i-1} \leq \underline{\omega}_i;\ \Omega_i, \Omega_{i+1} \neq \emptyset \to \underline{\omega}_i \leq \overline{\omega}_{i+1} \\ \omega &\in (\underline{\omega}_i, \overline{\omega}_i) \to \ \omega \notin \bigcup_{p \neq i} \Omega_p \end{aligned} \tag{19}$$

$$\Omega_{i-1} \cap \Omega_i \neq 0 \to \overline{\omega}_{i-1} = \underline{\omega}_i;\ a_{i-1} < a^E_{\underline{\omega}_i} < a_i \tag{20}$$

Finally, given A_M, let $\{\tilde{\varphi}(m \mid \omega)\}$ denote the message rule that maximizes agent E's payoff. From (19) – (20):

$$\begin{aligned} \omega &\notin \Omega_i \to \int_{M_i} \tilde{\varphi}(m \mid \omega) dm = 0;\ \omega \notin \bigcup_{p \neq i} \Omega_p \to \int_{M_i} \tilde{\varphi}(m \mid \omega) dm = 1 \\ \omega &\in \Omega_{i-1} \cap \Omega_i \to \int_{M_{i-1} \cup M_i} \tilde{\varphi}(m \mid \omega) dm = 1 \end{aligned} \tag{21}$$

Step 3. Let:

$$L_i = \{m \mid m \in M_i \ \text{ and } \tilde{\varphi}\,(m \mid \omega) > 0 \ \text{ for some } \omega \in \Omega_i\}$$

From (9) and (19) – (21) a Perfect Bayesian equilibrium must be such that:

$$\begin{aligned} &|\Omega^*_i| > 1 \to L^*_i \neq \emptyset;\ \int_{\underline{\omega}_i}^{\overline{\omega}_i} \rho^*\,(\omega \mid m \in L^*_i)\, d\omega = 1;\ a^*_i \in \left(a^N_{\underline{\omega}_i}, a^N_{\overline{\omega}_i}\right) \\ &\Omega^*_i = \omega \notin \bigcup_{p \neq i} \Omega^*_p \quad \to L^*_i \neq \emptyset;\ \rho^*\,(\omega \mid m \in L^*_i) = 1;\ a^*_i = a^N_\omega \end{aligned} \tag{22}$$

Step 4. From Assumption (2), since $a^E_0 < a^N_0$, given (12), $0 \in \Omega^*_1$ in equilibrium.

Suppose that in equilibrium some Ω^*_i is such that $\Omega^*_i = \breve{\omega} \notin \bigcup_{p \neq i} \Omega^*_p$. From (22), $a^E_{\breve{\omega}} < a^*_i = a^N_{\breve{\omega}}$. If $\breve{\omega} < 1$. given (19) and continuity, then $\Omega^*_{i+1} \neq \emptyset$ and there will a ω'' such that $a^E_{\breve{\omega}} < a^E_{\omega''} \leq a^*_i < a^*_{i+1}$; but, from (10) and (19), if $a^E_{\breve{\omega}} < a^E_{\omega''}$, then $\omega'' > \breve{\omega}$ and $\omega'' \notin \Omega^*_j$ with $j \leq i$, while, if $a^E_{\omega''} \leq a^*_i < a^*_{i+1}$, then $\omega'' \notin \Omega^*_j$ with $j \geq i+1$, that is contradictory. Instead, if $\breve{\omega} = 1$, given (16) and continuity, then $\Omega^*_{i-1} \neq \emptyset$ and $\left[U^E\,(a^*_i, \breve{\omega}, b) - U^E\left(a^*_{i-1}, \breve{\omega}, b\right)\right]$ will be strictly positive, while $\left[U^E\,(a^*_i, \overline{\omega}_{i-1}, b) - U^E\left(a^*_{i-1}, \overline{\omega}_{i-1}, b\right)\right]$ will be strictly negative:

there will be a ω' such that $U^E(a_i^*, \omega', b) = U^E\left(a_{i-1}^*, \omega', b\right)$ and $\omega' \in \Omega_{i-1}^* \cap \Omega_i^*$ so $|\Omega_i^*| > 1$ that is contradictory.

Now suppose that in equilibrium some Ω_i^* is such that $\Omega_i^* = \breve{\omega} = \Omega_{i+1}^*$. If $\breve{\omega} < 1$, from (20) and (18), then $a_{\breve{\omega}}^E \in \left(a_i^*, a_{i+1}^*\right)$ and $\Omega_{i+2}^* \neq \emptyset$. However, given (10) and continuity, there will be a ω'' such that $a_{\breve{\omega}}^E < a_{\omega''}^E \leq a_{i+1}^* < a_{i+2}^*$, $\omega'' > \breve{\omega}$ and $\omega'' \notin \Omega_j^*$ with $j \geq i+2$, that is contradictory. Instead, if $\breve{\omega} = 1$, then $a_1^N \in \left\{a_i^*, a_{i+1}^*\right\}$. If $a_{i+1}^* = a_1^N$, then $a_1^E \in \left(a_i^*, a_{i+1}^*\right)$ and there will be a ω' such that $a_{\omega'}^E \in \left[a_i^*, a_1^E\right)$, $\omega' < 1$ and $\omega' \notin \Omega_j^*$ with $j < i$, that is contradictory. If $a_i^* = a_1^N$, then $\Omega_{i+1}^* = \emptyset$, that is contradictory.

Suppose that $\Omega_i^* = \breve{\omega}$ and $\Omega_{i+1}^* = [\breve{\omega}, \overline{\omega}_{i+1}]$ with $\breve{\omega} < \overline{\omega}_{i+1}$. Since $0 \in \Omega_1^*$, then $\breve{\omega} \neq \{0,1\}$. Given (14), $\breve{\omega} \notin \Omega_{i-1}^*$, $a_{\breve{\omega}}^E \in \left(a_i^*, a_{i+1}^*\right)$ and there will be a ω' such that $a_{\omega'}^E \in \left[a_i^*, a_{\breve{\omega}}^E\right)$, $\omega' < \breve{\omega}$ and $\omega' \notin \Omega_j^*$ with $j < i$, that is contradictory.

Hence, given (13), a Perfect Bayesian equilibrium must be such that:

$$
\begin{aligned}
\exists \Omega_i^* &\;:\; \Omega_i^* \supseteq 1 \text{ and } i \in \{I^*-1, I^*\} \\
\left|\Omega_j^*\right| &> 1 \quad \vee j \in \{1, ..., I^*-1\} \\
\overline{\omega}_{j-1}^* &= \underline{\omega}_j^* \;\; \vee j \in \{1, ..., I^*\} \\
I^* &< \infty
\end{aligned}
$$

Step 5. There always exists a Perfect Bayesian Equilibrium with a unique action level played with probability one. Suppose that there exists an equilibrium partition with cardinality I^* greater than 1 and:

$$
\begin{aligned}
a_{i-1}^* \text{ and } \Omega_{i-1}^* &= [\overline{\omega}_{i-2}, \overline{\omega}_{i-1}]\,;\, a_i^* \text{ and } \Omega_i^* = [\overline{\omega}_{i-1}, \overline{\omega}_i] \\
a_{i+1}^* \text{ and } \Omega_{i+1}^* &= [\overline{\omega}_i, \overline{\omega}_{i+1}] \qquad 2 \leq i < I^*
\end{aligned}
$$

Given (16), if $\omega \in (\overline{\omega}_{i-1}, \overline{\omega}_i)$, then:

$$
\begin{aligned}
U^E(a_i^*, \omega, b) - U^E\left(a_{i-1}^*, \omega, b\right) &> U^E(a_i^*, \overline{\omega}_{i-1}, b) - U^E\left(a_{i-1}^*, \overline{\omega}_{i-1}, b\right) = \\
U^E\left(a_{i+1}^*, \overline{\omega}_i, b\right) - U^E(a_i^*, \overline{\omega}_i, b) &= 0 > U^E\left(a_{i+1}^*, \omega, b\right) - U^E(a_i^*, \omega, b)
\end{aligned}
$$

It follows that $U^E(a, \omega, b)$ is first increasing and then decreasing from a_{i-1}^* to a_{i+1}^*. Hence, there will be couples of actions, $(\hat{a}_{i-1}, \hat{a}_i)$, such that $\hat{a}_{i-1} \in \left(a_{i-1}^*, a_i^*\right)$, $\hat{a}_i \in \left(a_i^*, a_{i+1}^*\right)$ and $\left[U^E(\hat{a}_i, \omega, b) - U^E(\hat{a}_{i-1}, \omega, b)\right] = 0$ for $\omega \in (\overline{\omega}_{i-1}, \overline{\omega}_i)$. Since there exists an equilibrium partition with cardinality I^*, then:

$$
\begin{aligned}
\exists m' &\in L_{i-1}^* \text{ with } \varphi^*(m' \mid \omega)\,;\, \exists m \in L_i^* \text{ with } \varphi^*(m \mid \omega) \\
\exists m'' &\in L_{i+1}^* \text{ with } \varphi^*(m'' \mid \omega)
\end{aligned}
$$

Consider all the partitions of the type:

$\left\{\Omega_1' = [0, \overline{\omega}_1'],\ \Omega_i' = [\underline{\omega}_i', \overline{\omega}_i'],\ \Omega_{I^*-1}' = \left[\underline{\omega}_{(I^*-1)}', 1\right]\right\}$

with cardinality (I^*-1), where $\overline{\omega}_1 < \overline{\omega}_1'$, $\underline{\omega}_i < \underline{\omega}_i' < \overline{\omega}_i < \overline{\omega}_i'$ and $\underline{\omega}_{(I^*-1)}' <$

$\underline{\omega}_{I^*}$. Suppose that:

$$\begin{aligned}
\exists \hat{m} &\in L_{i-1}^{'} : \varphi(\hat{m} \mid \omega) = \varphi^*(m' \mid \omega) \quad \vee\, \omega \in \left[\overline{\omega}'_{i-2}, \overline{\omega}_{i-1}\right] \\
\exists \hat{m} &\in L_{i-1}^{'} : \varphi(\hat{m} \mid \omega) = \varphi^*(m \mid \omega) \quad \vee\, \omega \in \left[\overline{\omega}_{i-1}, \overline{\omega}'_{i-1}\right] \\
\exists \check{m} &\in L_{i}^{'} : \varphi(\check{m} \mid \omega) = \varphi^*(m \mid \omega) \quad \vee\, \omega \in \left[\overline{\omega}'_{i-1}, \overline{\omega}_{i}\right] \\
\exists \check{m} &\in L_{i}^{'} : \varphi(\check{m} \mid \omega) = \varphi^*(m'' \mid \omega) \quad \vee\, \omega \in \left[\overline{\omega}_{i}, \overline{\omega}'_{i}\right]
\end{aligned}$$

From (9), given $\hat{a}_i = \arg\max \int_{\overline{\omega}'_{i-1}}^{\overline{\omega}'_i} U^N(a,\omega) \frac{\varphi(\check{m}|\omega) f(\omega)}{\int_{\overline{\omega}'_{i-1}}^{\overline{\omega}'_i} \varphi(\check{m}|t) f(t) dt} d\omega$, then $\hat{a}_{i-1} < a^*_{i-1} < \hat{a}_i < a^*_i$. Since $\hat{a}_i$ is a monotonic function in $\left[\overline{\omega}'_{i-1}, \overline{\omega}'_i\right]$, there will exist an equilibrium partition with cardinality $(I^* - 1)$. ■

Proof. of Corollary 1

From Proposition 1 point 3, if there exists a P^*_{F,I^*} with $I^* > 2$, there will exists an equilibrium $P^*_{F,2}$. Consider an equilibrium $P^*_{F,2}$. The following condition must hold:

$$U^E(a^*_2, \omega, b) > U^E(a^*_1, \omega, b) \quad \vee\, \omega \in (\underline{\omega}^*_1, 1] \tag{23}$$

Let $F_1(\omega \mid q)$ be the d.f. $F(\omega)$ conditional on $\Omega_1 = [0,q]$ and $F_2(\omega \mid q)$ be the d.f. $F(\omega)$ conditional on $\Omega_2 = [q,1]$. From (6), $a_{F_1}(q)$ and $a_{F_2}(q)$ will be unique for every q. From (22), $a^N_0 = a_{F_1}(0)$. Hence, $a^N_0 < a_{F_1}(q) < a_F < a_{F_2}(q)$ for every $q \in (0,1)$. From strict concavity in (2), $\left[U^E(a_F,1,b) - U^E(a_{F_2}(q),1,b)\right]$ will be strictly positive if $\left[U^E(a_{F_1}(q),1,b) - U^E(a_F,1,b)\right]$ is nonnegative, while $\left[U^E(a_{F_1}(q),1,b) - U^E(a_F,1,b)\right]$ will be strictly positive if $\left[U^E\left(a^N_0,1,b\right) - U^E(a_F,1,b)\right]$ is nonnegative. Hence, if $\left[U^E\left(a^N_0,1,b\right) - U^E(a_F,1,b)\right]$ is nonnegative, then $U^E(a_{F_1}(q),1,b)$ will be greater than $U^E(a_F,1,b)$, that is greater than $U^E(a_{F_2}(q),1,b)$. Given increasing differences in (16), $U^E(a_{F_1}(q),\omega,b)$ will be greater than $U^E(a_{F_2}(q),\omega,b)$ for every $\omega \in \Omega$, in contradiction with (23). ■

Proof. of Lemma 1

$$\begin{aligned}
E_H[\omega] &= E_K[\omega] \qquad\qquad (24)\\
&\rightarrow \int_\Omega \omega\,[dH(\omega) - dK(\omega)] = -\int_\Omega [H(\omega) - K(\omega)]\,d\omega = 0
\end{aligned}$$

$$\begin{aligned}
\int_\Omega U^N_a(a,\omega)\,[dH(\omega) - dK(\omega)] &= \\
-\int_\Omega U^N_{a\omega}(a,\omega)\,[H(\omega) - K(\omega)]\,d\omega &= -U^N_{a\omega}(a,\overline{\omega}) \int_\Omega [H(\omega) - K(\omega)]\,d\omega + \\
+\int_\Omega U^N_{a\omega\omega}(a,\omega)\left[\int_{\underline{\omega}}^{\omega} [H(t) - K(t)]\,dt\right] d\omega &= 0
\end{aligned}$$

■

Proof. of Proposition 2

Suppose that under the d.f. $F(\omega)$ there exists a unique equilibrium with equilibrium actions $\left(a^*_{F_1}, a^*_{F_2}\right)$, and equilibrium $P^*_{F,2} = \{[0,\overline{\omega}^*_F], [\overline{\omega}^*_F, 1]\}$. Given

(6), from Corollary 1 it follows that $U^E(a_F, 1, b)$ is greater than $U^E\left(a_0^N, 1, b\right)$. Let $F_1(\omega)$ denote the d.f. $F(\omega)$ conditional on ω in $[0, \overline{\omega}_F^*]$, and $F_2(\omega)$ denote the d.f. $F(\omega)$ conditional on ω in $[\overline{\omega}_F^*, 1]$. Given strict concavity in (2), $a_{F_1}^* \in \left(a_0^N, a_F\right)$ and Corollary 1, it follows that $U^E\left(a_{F_2}^*, 1, b\right)$ is greater than $U^E\left(a_{F_1}^*, 1, b\right)$, that is greater than $U^E\left(a_0^N, 1, b\right)$.

Consider the set Ψ of distribution functions $G(\omega)$ with continuous density functions $g(\omega)$ such that (g/f) is decreasing on Ω (MLRP, (Karlin-Rubin (1956)). Hence, each $G(\omega)$ is dominated by $F(\omega)$ in the sense of first order stochastic dominance. Given MLRP:

$$\frac{\int_0^x g(\omega)\,d\omega}{\int_0^k g(\omega)\,d\omega} \geq \frac{\int_0^x f(\omega)\,d\omega}{\int_0^k f(\omega)\,d\omega} \qquad \vee\, x \leq k,\; k \in \Omega \tag{25}$$

There will be some d.f. $\tilde{G}(\omega)$ in Ψ such that the expected ω under $\tilde{G}(\omega)$, $E_{\tilde{G}}[\omega]$, is greater than the expected ω under $F_1(\omega)$, $E_{F_1}[\omega]$. Hence, $a_{\tilde{G}} \in \left(a_{F_1}^*, a_F\right)$. Given strict concavity in (2), under the d.f. $\tilde{G}$ the necessary condition stated by Corollary 1 will be satisfied. Hence, under the d.f. $\tilde{G}$ there will exists a unique equilibrium with equilibrium actions $\left(a_{\tilde{G}_1}^*, a_{\tilde{G}_2}^*\right)$, and equilibrium $P_{\tilde{G},2}^* = \left\{\left[0, \overline{\omega}_{\tilde{G}}^*\right], \left[\overline{\omega}_{\tilde{G}}^*, 1\right]\right\}$. Moreover, given (25), $\overline{\omega}_{\tilde{G}}^*$ will be lower than $\overline{\omega}_F^*$. Hence:

$$E_{\tilde{G}_1}[\omega] < E_{F_1}[\omega] < E_{\tilde{G}_2}[\omega] < E_{F_2}[\omega]$$

Let:

$$\Omega_1 = \left[0, \overline{\omega}_{\tilde{G}}^*\right] \qquad \Omega_2 = \left[\overline{\omega}_{\tilde{G}}^*, \overline{\omega}_F^*\right] \qquad \Omega_3 = \left[\overline{\omega}_F^*, 1\right]$$

and let $H_1(\omega)$ denote a d.f. on Ω_1, $H_2(\omega)$ a d.f. on Ω_2 and $H_3(\omega)$ a d.f. on Ω_3 such that:

$$E_{H_1}[\omega] = E_{\tilde{G}_1}[\omega]\,;\; E_{H_2}[\omega] = \tilde{\omega} \quad \text{with } \tilde{\omega} \in \tilde{\Omega} \cap\, \Omega_2;\; E_{H_3}[\omega] = E_{F_2}[\omega]$$

where $\tilde{\Omega} = \left(E_{F_1}[\omega], E_{\tilde{G}_2}[\omega]\right)$. Consider the following d.f. $Y(\omega)$:

$$\begin{aligned} Y(\omega) &= \alpha_1 H_1(\omega) \quad \vee\, \omega \in \Omega_1;\; Y(\omega) = \alpha_1 + \alpha_2 H_2(\omega) \quad \vee\, \omega \in \Omega_2 \\ Y(\omega) &= \alpha_1 + \alpha_2 + \alpha_3 H_3(\omega) \quad \vee\, \omega \in \Omega_3 \end{aligned}$$

where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. The d.f. $Y(\omega)$ is a mixture of the d.f. $H_i(\omega)$. Provided:

$$\begin{aligned} \alpha_1 &= \left\{E_{F_2}[\omega] - E_{\tilde{G}_2}[\omega]\right\}\left\{\tilde{\omega} - E_{F_1}[\omega]\right\}/\Delta \\ \alpha_2 &= \left\{E_{F_2}[\omega] - E_{\tilde{G}_2}[\omega]\right\}\left\{E_{F_1}[\omega] - E_{\tilde{G}_1}[\omega]\right\}/\Delta \\ \alpha_3 &= \left\{E_{F_1}[\omega] - E_{\tilde{G}_1}[\omega]\right\}\left\{E_{\tilde{G}_2}[\omega] - \tilde{\omega}\right\}/\Delta \\ \Delta &= \left\{E_{F_2}[\omega] - E_{\tilde{G}_2}[\omega]\right\}\left\{\tilde{\omega} - E_{F_1}[\omega]\right\} + \left\{E_{F_1}[\omega] - E_{\tilde{G}_1}[\omega]\right\}\left\{E_{F_2}[\omega] - \tilde{\omega}\right\} \end{aligned}$$

then:

$$\begin{aligned} E_Y[\omega \mid \omega \in \Omega_1] &= E_{\tilde{G}_1}[\omega]\,;\; E_Y[\omega \mid \omega \in \Omega_1 \cup \Omega_2] = E_{F_1}[\omega] \\ E_Y[\omega \mid \omega \in \Omega_2 \cup \Omega_3] &= E_{\tilde{G}_2}[\omega]\,;\; E_Y[\omega \mid \omega \in \Omega_3] = E_{F_2}[\omega] \end{aligned}$$

Given Lemma 1, under the d.f. $Y(\omega)$ both $P^*_{F,2}$ and $P^*_{\tilde{G},2}$ will be equilibrium partitions. ■

Proof. of Proposition 3

Under Assumption 2, let $F_i(\omega)$ be the d.f. $F(\omega)$ conditional on $\Omega_i = [\underline{\omega}_i, \overline{\omega}_i]$ with $\underline{\omega}_i < \overline{\omega}_i$, and $f_i(\omega)$ be its density function, continuous at each point of Ω_i. Then, $[\omega f_i(\omega)]$ is integrable on Ω_i, and $f_i(\omega)$ is positive for every $\omega \in (\underline{\omega}_i, \overline{\omega}_i)$. Let $\tilde{\omega} = E_{F_i}[\omega]$. Choose a ω' in $(\underline{\omega}_i, \tilde{\omega})$. Then $\left[\int_{\omega'}^{\omega} tf_i(t)\,dt / \int_{\omega'}^{\omega} f_i(t)\,dt\right]$ is continuous on $[\omega', \overline{\omega}_i]$ since both $\int_{\omega'}^{\omega} tf_i(t)\,dt$ and $\int_{\omega'}^{\omega} f_i(t)\,dt$ are continuous at each point of Ω_i. Choose a $\breve{\omega}$ in $(\omega', \overline{\omega}_i)$ such that $\left[\int_{\omega'}^{\breve{\omega}} tf_i(t)\,dt / \int_{\omega'}^{\breve{\omega}} f_i(t)\,dt\right]$ is lower than $\tilde{\omega}$. Given:

$$\begin{aligned} \Omega_A &= [\underline{\omega}, \overline{\omega}_a] \quad \Omega_B = [\underline{\omega}, \overline{\omega}_b] \quad \text{with } \overline{\omega}_b > \overline{\omega}_a > \underline{\omega} \\ \Omega_C &= [\underline{\omega}_c, \overline{\omega}] \quad \Omega_D = [\underline{\omega}_d, \overline{\omega}] \quad \text{with } \overline{\omega} > \underline{\omega}_d > \underline{\omega}_c \end{aligned}$$

let:

$$\begin{aligned} H_i(\omega) &= 0 \ \vee \omega < \underline{\omega};\ H_i(\omega) = \frac{\int_{\underline{\omega}}^{\omega} f_i(t)\,dt}{\int_{\Omega_i} f_i(t)\,dt} \quad \vee \omega \in \Omega_i;\ H_i(\omega) = 1 \ \vee \omega > \overline{\omega}_i \\ i &= A, B \end{aligned}$$

$$\begin{aligned} H_i(\omega) &= 0 \quad \vee \omega < \underline{\omega}_i;\ H_i(\omega) = \frac{\int_{\underline{\omega}_i}^{\omega} f_i(t)\,dt}{\int_{\Omega_i} f_i(t)\,dt} \quad \vee \omega \in \Omega_i; H_i(\omega) = 1 \ \vee \omega > \overline{\omega} \\ i &= C, D \end{aligned}$$

$H_B(\omega)$ dominates $H_A(\omega)$, and $H_D(\omega)$ dominates $H_C(\omega)$ in the sense of first order stochastic dominance. Hence:

$$\frac{\int_{\omega'}^{\breve{\omega}} tf_i(t)\,dt}{\int_{\omega'}^{\breve{\omega}} f_i(t)\,dt} < \tilde{\omega} < \frac{\int_{\omega'}^{\overline{\omega}} tf_i(t)\,dt}{\int_{\omega'}^{\overline{\omega}} f_i(t)\,dt}$$

From the intermediate value theorem for continuous functions, there exists a ω'' in $(\tilde{\omega}, \overline{\omega}_i)$ such that:

$$\frac{\int_{\omega'}^{\omega''} tf_i(t)\,dt}{\int_{\omega'}^{\omega''} f_i(t)\,dt} = \tilde{\omega} \tag{26}$$

Let:

$$\Omega_{iI} = [\underline{\omega}_i, \omega'),\ \Omega_{II} = [\omega', \omega''],\ \Omega_{III} = (\omega'', \overline{\omega_i}]$$

from (26) it follows that:

$$\int_{\Omega_i} tf_i(t)\,dt = \frac{\int_{\Omega_{iII}} tf_i(t)\,dt}{\int_{\Omega i_{II}} f_i(t)\,dt} \rightarrow \frac{\int_{\Omega i_I \cup \Omega_{iIII}} tf_i(t)\,dt}{\int_{\Omega_{iI} \cup \Omega_{iIII}} f_i(t)\,dt} = \frac{\int_{\Omega_{iII}} tf_i(t)\,dt}{\int_{\Omega i_{II}} f_i(t)\,dt} \tag{27}$$

Consider the following probability density function $f_i^{'}(\omega)$:

$$f_i^i(\omega) = \frac{pf_i(\omega)}{p\int_{\Omega_{iI}\cup\Omega_{iIII}} f_i(t)\,dt + q\int_{\Omega_{iii}} f_i(t)\,dt} \quad \vee\,\omega \in \Omega_{iI}\cup\Omega_{iIII} \tag{28}$$

$$f_i^i(\omega) = \frac{qf_i(\omega)}{p\int_{\Omega_{iI}\cup\Omega_{iIII}} f_i(t)\,dt + q\int_{\Omega_{iII}} f_i(t)\,dt} \quad \vee\,\omega \in \Omega_{iII}; \qquad p > q$$

From (27):

$$\int_{\Omega_i} tf_i(t)\,dt = \int_{\Omega_i} tf_i^i(t)\,dt \tag{29}$$

From (28), the d.f. $F_i'(\omega)$ will be greater than the d.f. $F(\omega)$ for every $\omega \in \Omega_{iI}$; instead, the d.f. $F_i(\omega)$ will be greater than the d.f. $F_i'(\omega)$ for every $\omega \in \Omega_{iIII}$. Moreover, there will be a $\hat{\omega}$ in (ω', ω'') such that the d.f. $F_i'(\omega)$ is greater than $F_i(\omega)$ for every $\omega \in [\omega', \hat{\omega})$, while the d.f. $F_i(\omega)$ is greater than the d.f. $F_i'(\omega)$ for every $\omega \in (\hat{\omega}, \omega'']$. From (29) and (24), given the single crossing property between the distribution functions (Diamond-Stiglitz (1974)), $F_i\prime(\omega)$ will be a mean preserving spread of $F_i(\omega)$.

Suppose that under $F(\omega)$ there exists a Perfect Bayesian equilibrium such that Ω_i belongs to the equilibrium partition. From Proposition 1, given ω in Ω_i, a constant $\varphi^*(m \mid \omega)$ for every m in M_i is an equilibrium message rule. Given Lemma 1 and m, m' in M_i, the message rule:

$\{\acute{\varphi}^*(m \mid \omega) = p\varphi^*(m \mid \omega), \acute{\varphi}^*(m' \mid \omega) = (1-p)\varphi^*(m \mid \omega)\} \vee\omega \in (\Omega_{iI}\cup\Omega_{iIII})$

$\{\acute{\varphi}^*(m \mid \omega) = q\varphi^*(m \mid \omega), \acute{\varphi}^*(m' \mid \omega) = (1-q)\varphi^*(m \mid \omega)\ \} \vee\omega \in \Omega_{iII}$

will be an equilibrium message rule as well. ■

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