Year 2009 *Paper* 52

Inequality Probabilities for Folded Normal Random Variables

John D. Cook*

*M. D. Anderson Cancer Center, cook@mdanderson.org

Inequality probabilities for folded normal random variables

John D. Cook
Department of Biostatistics
P. O. Box 301402, Unit 1409
The University of Texas, M. D. Anderson Cancer Center
Houston, Texas 77230-1402, USA
cook@mdanderson.org

July 11, 2009

Abstract

This note explains how to calculate the probability

$$\Pr(|X| > |Y|) \tag{1}$$

for normal random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. A random variable formed by taking the absolute value of a normal random variable is known as a *folded* normal random variable.

When $\sigma_X = \sigma_Y$, (1) can be evaluated simply using Equation (3) below. When $\sigma_X \neq \sigma_Y$, (1) can be reduced to a well-known problem using Equation (4).

1 Removing absolute values

To make the problem (1) easier to work with, we restate the problem in a form that does not involve absolute values. We begin by noting that the set of points

$$\{|x| > |y|\}$$

is bounded by the lines $x + y = 0$ and $x - y = 0$ and can thus be written as

$$\{x + y > 0 \ \wedge \ x - y > 0\} \cup \{x + y < 0 \ \wedge \ x - y < 0\}.$$

It follows that

$$\begin{aligned}\Pr(|X| > |Y|) &= \Pr(X + Y > 0 \ \wedge \ X - Y > 0) \\ &+ \Pr(X + Y < 0 \ \wedge \ X - Y < 0).\end{aligned}$$

Now define

$$\begin{aligned}U &= X + Y \\ V &= X - Y\end{aligned}$$

and so

$$\begin{aligned}U &\sim N(\mu_U, \sigma_U^2) \\ V &\sim N(\mu_V, \sigma_V^2)\end{aligned}$$

where $\mu_Y = \mu_x + \mu_y$, $\mu_U = \mu_x + \mu_y$, and $\sigma_U^2 = \sigma_V^2 = \sigma_X^2 + \sigma_Y^2$. We now have

$$\Pr(|X| > |Y|) = \Pr(U > 0 \ \wedge \ V > 0) + \Pr(U < 0 \ \wedge \ V < 0). \tag{2}$$

2 Joint probabilities

We now move on to calculating each of the inequalities on the right-hand side of Equation (2). First note that

$$\begin{aligned}\Pr(U > 0 \ \wedge \ V > 0) &= \Pr(U - \mu_U > -\mu_U \ \wedge \ V - \mu_V > -\mu_V) \\ &= \Pr\left(\frac{U - \mu_U}{\sigma_U} > -\frac{\mu_U}{\sigma_U} \ \wedge \ \frac{V - \mu_V}{\sigma_V} > -\frac{\mu_V}{\sigma_V}\right) \\ &= \Pr\left(Z_1 > -\frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 > -\frac{\mu_V}{\sigma_V}\right)\end{aligned}$$

where $Z_1 = (U - \mu_U)/\sigma_U$ and $Z_2 = (V - \mu_V)/\sigma_V$ are standard normal random variables. Similarly,

$$\Pr(U < 0 \ \wedge \ V < 0) = \Pr\left(Z_1 < \frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 < \frac{\mu_V}{\sigma_V}\right).$$

The random variables Z_1 and Z_2 have correlation

$$\frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}.$$

When $\sigma_X^2 = \sigma_Y^2$, Z_1 and Z_2 are uncorrelated and we have

$$\Pr(|X| > |Y|) = \Phi\left(-\frac{\mu_U}{\sigma_U}\right)\Phi\left(-\frac{\mu_V}{\sigma_V}\right) + \Phi\left(\frac{\mu_U}{\sigma_U}\right)\Phi\left(\frac{\mu_V}{\sigma_V}\right) \qquad (3)$$

where $\Phi(x)$ is the CDF of a standard normal random variable.

When $\sigma_X^2 \neq \sigma_Y^2$, Equation (3) does not hold. However, in that case we may still evaluate $\Pr(|X| > |Y|)$ as

$$\Pr\left(Z_1 > -\frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 > -\frac{\mu_V}{\sigma_V}\right) + \Pr\left(Z_1 < \frac{\mu_U}{\sigma_U} \ \wedge \ Z_2 < \frac{\mu_V}{\sigma_V}\right). \qquad (4)$$

Expression (4) cannot be evaluated in closed form. However, it does reduce to a known problem: evaluating rectangular probabilities for a bivariate normal random variable. These can be reduced to a one-dimensional integral that can be evaluated numerically. See "Numerical Computation of Rectangular Bivariate and Trivariate Normal and t Probabilities" by Alan Genz, Statistics and Computing, 14 (2004), pp. 151-160.